nor.1 Segments and Cuts

pt:nor:seg:sec An introduction rule followed by an elimination rule is a detour in a derivation, and to normalize a derivation is to eliminate such detours. However, the ∨Elim and ∃Elim make even the definition of "detour we want to eliminate" a bit complicated. The problem is this. Because of these rules, it's not guaranteed that in a detour the elimination rule *immediately* follows the introduction rule. There could be an ∨Elim or ∃Elim in between, e.g.,

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \exists x\,\chi(x)\end{matrix} \qquad \dfrac{\begin{matrix}[\varphi]^x\\ \vdots\\ \psi\end{matrix}}{\varphi\to\psi}\,{\to}\text{Intro}}{\varphi\to\psi}\,{\exists}\text{Elim} \qquad \begin{matrix}\vdots\\ \varphi\end{matrix}}{\psi}\,{\to}\text{Elim}$$

In fact, any number of ∨Elim or ∃Elim can separate the →Intro from the →Elim, e.g.,

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \exists x\,\chi(x)\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ \theta\vee\alpha\end{matrix} \qquad \dfrac{\begin{matrix}[\varphi]^x\\ \vdots\\ \psi\end{matrix}}{\varphi\to\psi}\,{\to}\text{Intro} \qquad \dfrac{\begin{matrix}[\varphi]^x\\ \vdots\\ \psi\end{matrix}}{\varphi\to\psi}\,{\to}\text{Intro}}{\varphi\to\psi}\,{\vee}\text{Elim}}{\varphi\to\psi}\,{\exists}\text{Elim} \qquad \begin{matrix}\vdots\\ \varphi\end{matrix}}{\psi}\,{\to}\text{Elim}$$

This example shows that the same →Elim might be involved in multiple detours, even, as it follows more than one →Intro. To account for all these possibilities, we define detours using certain kinds of sequences of formula occurrences in an **N1i**-proof called *segments*. In our example, it's a sequence of occurrences of $\varphi\to\psi$, where a minor premise of ∨Elim or ∃Elim follows its conclusion. In the example, there are two such segments, each containing three occurrences of $\varphi\to\psi$. Here is the official definition.

Definition nor.1. A *segment* in a **N1i** derivation δ is a sequence of occurrences $\varphi_1, \dots, \varphi_n$ of the same formula φ in δ where

1. φ_1 is not the conclusion of ∨Elim or ∃Elim;
2. φ_n is not a minor premise of ∨Elim or ∃Elim;
3. For $i = 1, \dots, n-1$, φ_i is a minor premise of ∨Elim or ∃Elim and A_{i+1} is the conclusion of that inference.

The *length* of that segment is n.

Not every segment is a target for elimination (a detour). The ones that are are called *cuts*.

Definition nor.2. A *cut* is a segment in which φ_n is the major premise of an elimination inference, and either $n = 1$ and $\varphi_1 = \varphi_n$ is the conclusion of an introduction inference or of $\bot_I$, or $n > 1$.

Note that a cut segment need not begin with the conclusion of an introduction rule. All that's required is that it ends with the major premise of an elimination rule. However, a segment of length 1 does have to be the conclusions of introduction rules to count as a cut.

Definition nor.3. The *rank* of a cut is $\mathrm{dp}(\varphi)$. The *cut rank* $\mathrm{cr}(\delta)$ of a derivation δ is the maximal rank of cuts in δ. A cut is *maximal* in δ if its rank is $\mathrm{cr}(\delta)$, i.e., it has maximal rank. The *cut length* of δ is the sum of the lengths of all its maximal cuts.

Photo Credits

Bibliography