nor.1 Reduction Conversions

pt:nor:red: sec

If a cut in a derivation in **N1i** has length 1, it is a segment consisting of a single formula occurrence, and that formula is both the conclusion of an introduction inference and the major premise of an elimination inference. In the normalization proof, such cuts are "reduced" by replacing the sub-derivation ending in such an introduction/elimination pair by a new derivation in which the pair of inferences has been eliminated. Such a replacement may result in new cuts. However, we can convince ourselves that the new cuts have lower rank than the cut being reduced. Thus, the new derivation either has lower cut length (because a maximal cut of length 1 has been removed) or the cut rank of the derivation has been reduced (if the maximal cut in question was the only maximal cut).

For instance, if the inferences involved are →Intro and →Elim, $\varphi \equiv \psi \to \chi$ and there is a sub-derivation δ_1 that ends in:

$$\dfrac{\dfrac{\begin{array}{c}[\psi]^x\\ \vdots\,\delta_2\\ \chi\end{array}}{\psi \to \chi}\,{\scriptstyle x}\,{\to}\text{Intro} \qquad \begin{array}{c}\vdots\,\delta_3\\ \psi\end{array}}{\chi}\,{\to}\text{Elim}$$

To remove this cut we replace the subproof δ_1 in δ by the following:

$$\begin{array}{c}\vdots\,\delta_3\\ \psi\\ \vdots\,\delta_2[\delta_3/\psi^x]\\ \chi\end{array}$$

By $\delta_2[\delta_3/\psi^x]$ we mean the derivation that results from δ_2 by replacing every assumption ψ that's labelled x with the entire derivation δ_3. This derivation no longer has open assumptions of the form ψ with label x. It has no other assumptions labelled x, since we assume, as usual, that no two formulas occur as assumptions with the same label. Hence, the new derivation has the same open assumptions as δ_1, every inference in δ_2 that discharges assumptions discharges the same assumptions in $\delta_2[\delta_3/\psi^x]$, and every inference in δ_3 that discharges assumptions results in possibly multiple copies, which discharge the corresponding assumptions in $\delta_2[\delta_3/\psi^x]$.

Any cut that lies entirely within δ_2, δ_3, or δ_4, or entirely outside δ_1 in δ is unchanged: it still involves the same formulas, goes through the same inferences, so in particular has the same rank and length as the corresponding cut in δ. We have removed a maximal cut of length 1 and so either have reduced the cut length, or reduced the cut rank (if the cut length of δ was 1 and this was the only maximal cut).

However, two things may happen:

1. By replacing all assumptions ψ^x by δ_3, we have multiplied all the cuts in δ_3. If some of those were maximal cuts, the cut length of the new derivation may be greater than the cut length of the old derivation. We have to guarantee that this does not happen. We do so by applying reduction conversions only to *rightmost* maximal cuts, that is, to cuts which are such that no maximal cut lies on a branch through δ to the right of the cut being reduced. If there were a cut in δ_3 it would be to the right of the cut being reduced here, and so we wouldn't be working on a rightmost cut. By always selecting rightmost cuts, we guarantee that either we decrease the cut length of the derivation, or even reduce the cut rank.

2. If an assumption ψ^x in δ_2 is the major premise of an elimination inference, and the last inference of δ_3 is $\vee$Elim, $\exists$Elim, or an introduction inference, we have generated a new cut by grafting δ_3 onto δ_2 at such an assumption. What was just an assumption in δ_2 is now a cut of length 1 (the conclusion of an introduction inference and major premise of an elimination inference) or the last formula occurrence of a cut segment (major premise of an elimination inference and conclusion of $\vee$Elim or $\exists$Elim). If B^x was a minor premise of $\vee$Elim or $\exists$Elim, it already was the first formula occurrence of a segment, and possibly of a cut segment. In the new derivation that segment may now be longer. However, observe that the formula making up this new cut or existing cut segment is ψ, and $\mathrm{dp}(\psi) < \mathrm{dp}(\psi \to \chi)$. Hence, although we have potentially added cuts or increased the length of cuts, none of these are maximal cuts. Thus we have either decreased the cut rank, or, if there are cuts of the same rank remaining, have decreased the cut length.

The process of reducing cut of length 1 is called a *reduction conversion* (or *detour conversion*). The patterns for some of the rule pairs is given in Table 1. (The remaining are left as exercises.)

Problem nor.1. Give the reduction conversion for ¬Intro followed by ¬Elim.

There is one case still to be considered. The definition of cut includes the case where the length of the cut is 1 and φ is the conclusion of $\bot_I$ as well as the major premise of an elimination inference. This case is dealt with in a similar way as when φ is the conclusion of an introduction rule. For instance, replace

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \delta_2\\ \vdots\\ \bot\end{array}}{\psi \to \chi} \quad \begin{array}{c}\vdots\\ \delta_3\\ \vdots\\ \psi\end{array}}{\chi} \to\text{Elim} \quad \text{by} \quad \dfrac{\begin{array}{c}\vdots\\ \delta_2\\ \vdots\\ \bot\end{array}}{\chi}$$

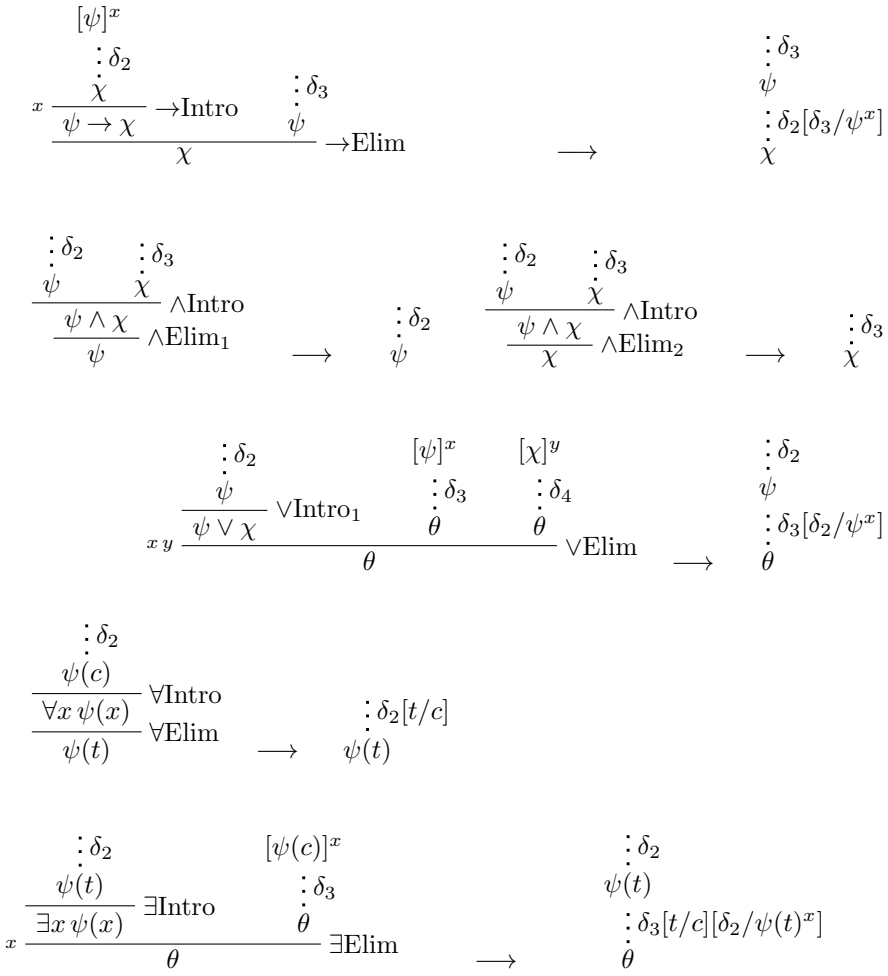
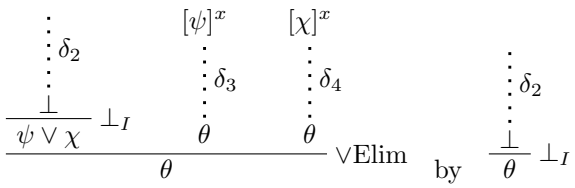


Table 1: Some reduction conversions for **N1i**.

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tab:reduction-conversions

We must be a little careful in the cases where $\varphi \equiv (\psi \vee \chi)$ or $\varphi \equiv \exists x\, \psi(x)$. E.g., if we replaced



we might be introducing a new cut with cut formula θ, and there is no guarantee that $\mathrm{dp}(\theta) < \mathrm{dp}(\psi \vee \chi)$! So we must proceed the same way here as we did for an $\vee$Intro/$\vee$Elim reduction and replace with

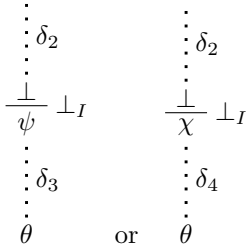


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Bibliography