nor.1 Permutation Conversions

pt:nor:per:sec Cuts in **N1i** are segments that end in the major premise of an elimination rule. One case in the normalization proof is to reduce the length of maximal cuts, and with it the cut length of the derivation. A cut of length > 1 can end in many ways depending on whether the last formula occurrence of φ in the cut is the conclusion of ($\vee$Elim or $\exists$Elim) and depending on the elimination inference it is the major premise of.

For instance, if the inferences involved are $\vee$Elim and $\wedge$Elim, $\varphi \equiv \psi \vee \chi$ and there is a sub-derivation δ_1 that ends in:

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\delta_2\\ \theta \vee \alpha\end{matrix} \qquad \begin{matrix}[\theta]^x\\ \vdots\,\delta_3\\ \psi \wedge \chi\end{matrix} \qquad \begin{matrix}[\alpha]^y\\ \vdots\,\delta_4\\ \psi \wedge \chi\end{matrix}}{\psi \wedge \chi}\,x\,y\ \vee\text{Elim}}{\psi}\ \wedge\text{Elim}_1$$

The last formula occurrence φ_n in the cut is the conclusion of $\vee$Elim, the lower $\psi \wedge \chi$. The second-to-last formula occurrence φ_{n-1} is one of the minor premises of $\vee$Elim.

To reduce the length of this cut, we can switch the order ("permute") of the $\vee$Elim and $\wedge$Elim inferences. That is, we replace the subproof δ_1 in δ by the following:

$$\dfrac{\begin{matrix}\vdots\,\delta_2\\ \theta \vee \alpha\end{matrix} \qquad \dfrac{\begin{matrix}[\theta]^x\\ \vdots\,\delta_3\\ \psi \wedge \chi\end{matrix}}{\psi}\ \wedge\text{Elim}_1 \qquad \dfrac{\begin{matrix}[\alpha]^y\\ \vdots\,\delta_4\\ \psi \wedge \chi\end{matrix}}{\psi}\ \wedge\text{Elim}_1}{\psi}\,x\,y\ \vee\text{Elim}$$

The cuts that ended in the premise of $\wedge$Elim below $\vee$Elim now end in the premise of one of the $\wedge$Elim inferences above the minor premises of $\vee$Elim, i.e., the length of each has decreased by 1.

Moreover, any cut that lies entirely within δ_2, δ_3, or δ_4, or entirely outside δ_1 in δ is unchanged: it still involves the same formulas, goes through the same inferences, so in particular has the same rank and length as the corresponding cut in δ. So the cut rank of the new derivation has not increased, and the cut length of the new derivation is less than that of the old derivation.

Problem nor.1. After a permutation conversion, the length of at least one cut is reduced by 1. If we permute $\wedge$Elim across an $\vee$Elim, there are actually two cut segments the length of which decrease, so in this case the cut length of the derivation decreases by at least 2. Can the cut length of a derivation decrease by more than 2 by a single such permutation? Can the cut *rank* decrease?

The process of moving an elimination rule above an ∨Elim or ∃Elim rule is called a *permutation conversion*. The patterns for some of the rule pairs is given in Table 1. (The remaining are left as exercises.)

$$\dfrac{\theta \vee \alpha \qquad \dfrac{\begin{array}{c}[\theta]^x \\ \vdots \\ \psi \wedge \chi\end{array} \qquad \begin{array}{c}[\alpha]^y \\ \vdots \\ \psi \wedge \chi\end{array}}{\dfrac{\psi \wedge \chi}{\psi}\,\wedge\mathrm{Elim}_1}\,\vee\mathrm{Elim}}{\psi} \quad x\ y \qquad \longrightarrow$$

$$\dfrac{\theta \vee \alpha \qquad \dfrac{\begin{array}{c}[\theta]^x \\ \vdots \\ \psi \wedge \chi\end{array}}{\dfrac{\psi \wedge \chi}{\psi}\,\wedge\mathrm{Elim}_1} \qquad \dfrac{\begin{array}{c}[\alpha]^y \\ \vdots \\ \psi \wedge \chi\end{array}}{\dfrac{\psi \wedge \chi}{\psi}\,\wedge\mathrm{Elim}_1}}{\psi}\,\vee\mathrm{Elim} \quad x\ y$$

$$\dfrac{\theta \vee \alpha \qquad \dfrac{\begin{array}{c}[\theta]^x \\ \vdots \\ \psi \to \chi\end{array} \qquad \begin{array}{c}[\alpha]^y \\ \vdots \\ \psi \to \chi\end{array}}{\dfrac{\psi \to \chi \qquad \psi}{\chi}\,\to\mathrm{Elim}}\,\vee\mathrm{Elim}}{\chi} \quad x\ y \qquad \longrightarrow$$

$$\dfrac{\theta \vee \alpha \qquad \dfrac{\begin{array}{c}[\theta]^x \\ \vdots \\ \psi \to \chi\end{array} \quad \psi}{\chi}\,\to\mathrm{Elim} \qquad \dfrac{\begin{array}{c}[\alpha]^y \\ \vdots \\ \psi \to \chi\end{array} \quad \psi}{\chi}\,\to\mathrm{Elim}}{\chi}\,\vee\mathrm{Elim} \quad x\ y$$

$$\dfrac{\theta \vee \alpha \qquad \dfrac{\begin{array}{c}[\theta]^x \\ \vdots \\ \forall x\,\psi(x)\end{array} \qquad \begin{array}{c}[\alpha]^y \\ \vdots \\ \forall x\,\psi(x)\end{array}}{\dfrac{\forall x\,\psi(x)}{\psi(t)}\,\forall\mathrm{Elim}}\,\vee\mathrm{Elim}}{\psi(t)} \quad x\ y \qquad \longrightarrow$$

$$\dfrac{\theta \vee \alpha \qquad \dfrac{\begin{array}{c}[\theta]^x \\ \vdots \\ \forall x\,\psi(x)\end{array}}{\dfrac{\forall x\,\psi(x)}{\psi(t)}\,\forall\mathrm{Elim}} \qquad \dfrac{\begin{array}{c}[\alpha]^y \\ \vdots \\ \forall x\,\psi(x)\end{array}}{\dfrac{\forall x\,\psi(x)}{\psi(t)}\,\forall\mathrm{Elim}}}{\psi(t)}\,\vee\mathrm{Elim} \quad x\ y$$

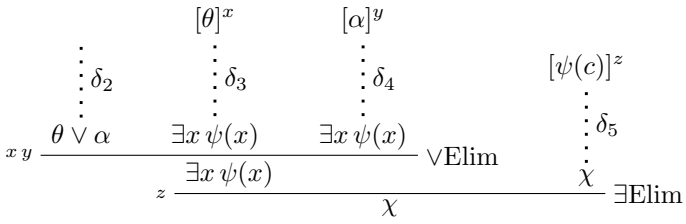
Table 1: Some permutation conversions for **N1i**.

pt:nor:per:
tab:permutation-conversions

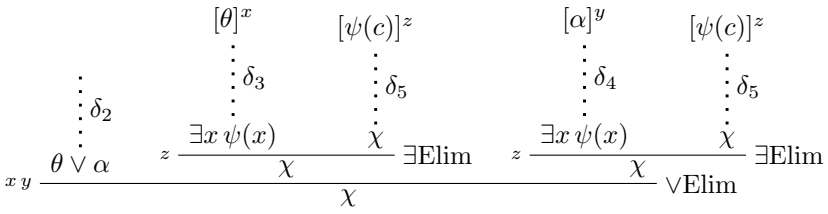
Problem nor.2. Give the permutation conversions for ∨Elim followed by ∨Elim or ¬Elim.

Problem nor.3. Give the permutation conversions for ∃Elim followed by ∃Elim. Explain why no eigenvariable condition is violated in the latter.

We should convince ourselves that when applying a permutation conversion, we don't perhaps violate an eigenvariable condition. Consider



If we replace this by



we might worry that the eigenvariable c might occur in, say, θ. After all, the assumption θ is discharged above the original $\exists$Elim and so in the original derivation it does not occur in an assumption that's open above the major premise of the $\exists$Elim inference, and so there the eigenvariable condition is satisfied. But in the new derivation, θ is not discharged until after the $\exists$Elim inferences, and so the eigenvariable condition would be violated. Recall, however, that we assume our derivations are regular: every eigenvariable is the eigenvariable of a single inference and only occurs above the minor premise of $\exists$Elim in the derivation. In particular, c cannot occur in δ_3 or δ_4.

This example shows something else: Note that the new derivation has two copies of δ_5. This means that if δ_5 contains a maximal cut, we have *not* reduced the cut length!Thus we must take care that we apply a permutation conversion only when we know that there is no maximal cut in δ_5. We will ensure this by always selecting a *topmost* maximal cut, that is a maximal cut such that the sub-derivation (in this case, δ_1) contains no maximal cut that ends above the last inference in the sub-derivation. This rules out the possibility that there are other maximal cuts in δ_5 (or indeed in δ_2, δ_3, or δ_4).

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Bibliography