nor.1 Introduction

pt:nor:int:sec

If you have derived a conditional $\varphi \to \psi$, you might be in a position to use it in a derivation of ψ by using $\to$Elim. This would of course also require a derivation δ_1 of φ. Your derivation δ_1 of $\varphi \to \psi$, more likely than not, ends in $\to$Intro, which turns a derivation δ_2 of ψ from the open assumption φ into one of $\varphi \to \psi$. If it does, your final derivation has the form

$$\dfrac{\dfrac{\begin{array}{c}[\varphi]^x \\ \vdots\,\delta_2 \\ \psi\end{array}}{\varphi \to \psi}\,{\scriptstyle x}\,\to\text{Intro} \qquad \begin{array}{c}\vdots\,\delta_2 \\ \varphi\end{array}}{\psi}\,\to\text{Elim}$$

Although your strategy is efficient in that you re-used something you already had (a derivation of $\varphi \to \psi$), your derivation isn't. It seems like a detour to introduce $\varphi \to \psi$ only to immediately eliminate it. There should be a more direct way to derive ψ.

In fact, there always is. "Detours" like the above—an introduction rule immediately followed by an elimination rule—can always be removed from natural deduction derivations. This is called *normalization* and the result is a *normal* derivation (or a derivation *in normal form*).

The proof of this result is, like the cut-elimination theorem in the sequent calculus, somewhat involved and requires us to consider lots of cases. It is harder to prove for classical **N1c** than it is for intuitionistic **N1i**. Just like cut-elimination shows that you can't derive more using the Cut rule in the sequent calculus, normalization shows that you can't prove more things using detours than you can while avoiding them. There are other similarities: normal derivation can be much longer than the shortest derivation with detours of the same result, just like derivations in the sequent calculus can be much shorter with cuts than the shortest cut-free derivation. The sub-formula property, that you can always derive a result using only sub-formulas of the result in your derivation, follows from normalization for natural deduction just as it follows from cut-elimination for the sequent calculus.

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Bibliography