nat.1 Translating from N2 to G2

pt:nat:tni:sec
pt:nat:tni:prop:N2-to-G2

Proposition nat.1. *If* **N2c** (**N2i**) $\vdash \Gamma \Rightarrow \varphi$ *then* **G2c** (**N2i**)+Cut $\vdash \Gamma' \Rightarrow \Delta$ *where* Γ' *is the multiset of formulas resulting from* Γ *by removing labels, and* $\Delta = \{\varphi\}$ *if* φ *is not* $\bot$*, and* $\Delta = \emptyset$ *if it is.*

Proof. The proof is again by induction on the height of the derivation δ of $\Gamma \Rightarrow \varphi$. If δ is an axiom $x : \varphi \Rightarrow \varphi$, π is the corresponding axiom $\varphi \Rightarrow \varphi$ or $\bot \Rightarrow$ if $\varphi \equiv \bot$.

If δ ends in an inference rule, we distinguish cases:

1. If R is $\to$Intro with principal formula $\psi \to \chi$ and $x : \psi$ is present in the context of the premise but not the conclusion, then δ ends in

$$\dfrac{\begin{array}{c}\vdots\,\delta_1\\ x : \psi, \Gamma \Rightarrow \chi\end{array}}{\Gamma \Rightarrow \psi \to \chi}\ \to\text{Intro}$$

   The inductive hypothesis yields a derivation π_1 of $\psi, \Gamma' \Rightarrow \chi$ (or of $\psi, \Gamma' \Rightarrow$ if $\chi \equiv \bot$), from which we obtain π by applying $\to$R (after applying WR in case $\chi \equiv \bot$).

2. If R is $\to$Intro, but $x : \psi$ is not present in the context of the premise, then δ ends in

$$\dfrac{\begin{array}{c}\vdots\,\delta_1\\ \Gamma \Rightarrow \chi\end{array}}{\Gamma \Rightarrow \psi \to \chi}\ \to\text{Intro}$$

   The inductive hypothesis yields a derivation π_1 of $\Gamma' \Rightarrow \chi$, from which we obtain π by applying WL with ψ, and then $\to$R.

3. If R is $\to$Elim, δ ends in

$$\dfrac{\begin{array}{cc}\vdots\,\delta_1 & \vdots\,\delta_2\\ \Gamma_1 \Rightarrow \psi \to \varphi & \Gamma_2 \Rightarrow \psi\end{array}}{\Gamma_1, \Gamma_2 \Rightarrow \varphi}\ \to\text{Elim}$$

   where $\Gamma = \Gamma_1 \cup \Gamma_2$. The inductive hypothesis yields derivations π_1 of $\Gamma_1' \Rightarrow \psi \to \varphi$ and π_2 of $\Gamma_2' \Rightarrow \psi$. We obtain the derivation π as

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1 & \dfrac{\begin{array}{c}\vdots\,\pi_2\\ \Gamma_2' \Rightarrow \psi\end{array} \qquad \varphi \Rightarrow \varphi}{\psi \to \varphi, \Gamma_2' \Rightarrow \varphi}\ \to\text{L}\\ \Gamma_1' \Rightarrow \psi \to \varphi & \end{array}}{\Gamma_1', \Gamma_2' \Rightarrow \varphi}\ \text{Cut}$$

The multiset $\Gamma_1' \cup \Gamma_2'$ may not equal $\Gamma' = (\Gamma_1 \cup \Gamma_2)'$: e.g., if both Γ_1 and Γ_2 contain $x : \chi$ then $\Gamma_1' \cup \Gamma_2'$ contains two copies of χ but $(\Gamma_1 \cup \Gamma_2)'$ contains only one copy. We can correct this by adding suitable CL inferences.

In the event that $\varphi \equiv \bot$, the inductive hypothesis applied to δ_2 yields a derivation of $\Gamma_2' \Rightarrow$, from which we can obtain the derivation π by adding suitable WL inferences and a WR with φ.

4. If R is $\bot_C$, δ ends in

$$\dfrac{\begin{array}{c}\vdots\\ \delta_1\\ \vdots\end{array}}{\begin{array}{c} x : \neg\varphi, \Gamma \Rightarrow \bot \\ \hline \Gamma \Rightarrow \varphi \end{array} \bot_C}$$

By inductive hypothesis, there is a derivation π_1 of $\neg\varphi, \Gamma' \Rightarrow$. We take the derivation π to be

$$\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\Rightarrow \varphi, \neg\varphi}\,\neg\mathrm{R} \qquad \begin{array}{c}\vdots\\ \pi_1\\ \vdots\end{array}}{\begin{array}{c} x : \neg\varphi, \Gamma' \Rightarrow \\ \hline \Gamma' \Rightarrow \varphi \end{array} \mathrm{Cut}}$$

Note that only the translation of $\bot_C$ results in a derivation π with more than one formula on the right. That is, the translation of a **N2i** derivation is a **G2i**-derivation. □

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Bibliography