## nat.1 Translating from G2i to N2i

pt:nat:tgi: sec

The antecedents of sequents in **G2i** are multisets $\Gamma$ of formulas, while those of sequents in **N2i** are sets $\Gamma'$ of labelled formulas with the property that every label occurs only once. Let's say a set $\Gamma'$ of labelled formulas *corresponds to* the multiset $\Gamma$ if, for every formula $\varphi$, the number of elements of $\Gamma'$ of the form $x : \varphi$ is less than or equal to the multiplicity of $\varphi$ (i.e., the number of copies of $\varphi$) in $\Gamma$.

pt:nat:tgi: prop:G2i-to-N2i

**Proposition nat.1.** *If* **G2i** + Cut $\vdash \Gamma \Rightarrow \Delta$ *then* **N2i** $\vdash \Gamma' \Rightarrow \chi$*, where* $\chi \equiv \bot$ *if* $\Delta$ *is empty and* $\Delta = \{\varphi\}$ *otherwise, and* $\Gamma'$ *corresponds to* $\Gamma$*.*

*Proof.* We show that if $\pi$ is a derivation of $\Gamma \Rightarrow \Delta$ in **G2ci**, then there is a $\Gamma'$ corresponding to $\Gamma$ and a derivation $\delta$ of $\Gamma' \Rightarrow \chi$ in **N2i**. We do this by induction on $n = \mathrm{ht}(\pi)$.

If $n = 0$, then $\pi$ is an axiom. If it is the axiom $\bot \Rightarrow$ , $\delta$ is the axiom $x : \bot \Rightarrow \bot$, i.e., $\Gamma' = \{x : \bot\}$ and $\chi \equiv \bot$. If $\pi$ is an axiom of the form $\varphi \Rightarrow \varphi$, then $\delta$ is $x : \varphi \Rightarrow \varphi$.

If $n > 0$, $\pi$ ends in a rule $R$. The inductive hypothesis guarantees that there are derivations of sequents corresponding to the premises of that rule in **N2c**. We can assume that all discharge labels in those proofs are distinct, and that a label $x$ only occurs above an inference that discharges it, if it is discharged at all (by relabelling if necessary). We then show how these derivations can be combined to provide proofs of a suitable $\Gamma' \Rightarrow \chi$. We distinguish cases according to $R$.

1. If $R$ is WR, $\pi$ ends in

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\\ \Gamma \Rightarrow\end{array}}{\Gamma \Rightarrow \varphi}\,\mathrm{WR}$$

   The induction hypothesis applied to $\pi_1$ yields a derivation of $\Gamma' \Rightarrow \bot$. We apply $\bot_I$ to get a derivation of $\Gamma' \Rightarrow \varphi$.

2. If $R$ is WL, $\pi$ ends in

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array}}{\varphi, \Gamma \Rightarrow \Delta}\,\mathrm{WR}$$

   The induction hypothesis applied to $\pi_1$ yields a derivation of $\Gamma' \Rightarrow \chi$. Which we take as $\delta$. Note that if $\Gamma'$ corresponds to $\Gamma$, it also corresponds to $\varphi, \Gamma$ as the number of $x : \varphi$ in $\Gamma'$ is no greater than the number of copies of $\varphi$ in $\Gamma$, and so also no greater than the number of copies of $\varphi$ in $\Gamma \cup \{\varphi\}$.

3. If $R$ is CL, $\pi$ ends in

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\end{array}}{\dfrac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\,\mathrm{CL}}$$

The induction hypothesis applied to $\pi_1$ yields a derivation of $\Pi, \Gamma' \Rightarrow \chi$, where $\Pi \subseteq \{x : \varphi, y : \varphi\}$. If $\Pi = \{x : \varphi, y : \varphi\}$, replace all labelled formulas $y : \varphi$ in $\pi_1$ by $x : \varphi$. Since $x$ occurs in the context of the end-sequent of $\pi_1$ we can assume that no inference in $\pi_1$ discharges a formula of the form $x : \psi$, i.e., no $x : \psi$ on the left side of a sequent in $\pi_1$ is active. Consequently, the result is still a derivation in **N2i**.

4. If $R$ is $\to$R, $\pi$ ends in

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\end{array}}{\dfrac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\,{\to}\mathrm{R}}$$

By the inductive hypothesis, there is a derivation $\delta_1$ of $x : \varphi, \Gamma' \Rightarrow \psi$ or $\Gamma' \Rightarrow \psi$ where $\Gamma'$ corresponds to $\Gamma$. We can take $\delta_1$ to be $\delta$ plus an application of $\to$Intro.

5. If $R$ is $\to$L, $\pi$ ends in

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\end{array}\quad\quad\begin{array}{c}\vdots\\ \pi_2\\ \vdots\end{array}}{\dfrac{\Gamma_1 \Rightarrow \varphi \qquad \psi, \Gamma_2 \Rightarrow \Delta}{\varphi \to \psi, \Gamma_1, \Gamma_2 \Rightarrow \Delta}\,{\to}\mathrm{L}}$$

The inductive hypothesis yields derivations $\delta_1$ of $\Gamma_1' \Rightarrow \varphi$ and $\delta_2$ of $x : \psi, \Gamma_2' \Rightarrow \chi$ or of of $\Gamma_2' \Rightarrow \chi$. If $\delta_2$ is of the latter form, $\delta_2$ can serve as $\delta$. Otherwise, we can relabel all formulas and discharge labels in $\pi_1$ to ensure that no label occurs in both $\delta_1$ and $\delta_2$. Then $\Gamma_1' \cup \Gamma_2'$ corresponds to $\Gamma_1 \cup \Gamma_2$. Since $x : \psi$ occurs in the end-sequent of $\delta_2$, $x : \psi$ cannot be active in any inference with discharge label $x$ in $\delta_2$. Replace every axiom $x : \psi \Rightarrow \psi$ in $\delta_2$ with the derivation $\delta_3$

$$\dfrac{x : \varphi \to \psi \Rightarrow \varphi \to \psi \qquad \begin{array}{c}\vdots\\ \delta_1\\ \vdots\end{array}\ \Gamma_1' \Rightarrow \varphi}{x : \varphi \to \psi \Rightarrow \psi}\,{\to}\mathrm{Elim}$$

and every occurrence of $x : \psi$ in $\delta_2$ by $x : \varphi \to \psi$. The result $\delta_2[\delta_3/x : \psi]$ is a derivation of $x : \varphi \to \psi, \Gamma_1', \Gamma_2' \Rightarrow \chi$.

6. If $R$ is $\wedge$R, $\pi$ ends in

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1 & \vdots\,\pi_2\\ \Gamma_1 \Rightarrow \varphi & \Gamma_2 \Rightarrow \psi\end{array}}{\Gamma_1, \Gamma_2 \Rightarrow \varphi \wedge \psi}\,\wedge\text{R}$$

The inductive hypothesis yields proof $\delta_1$ of $\Gamma_1' \Rightarrow \varphi$ and $\delta_2$ of $\Gamma_2' \Rightarrow \varphi$. By relabelling all formulas and discharge labels in $\pi_2$ we can ensure that no label occurs in both $\delta_1$ and $\delta_2$. Then $\Gamma_1' \cup \Gamma_2'$ corresponds to $\Gamma_1 \cup \Gamma_2$. We obtain a derivation $\delta$ by applying $\wedge$Intro to the end-sequents of $\delta_1$ and $\delta_2$.

7. If $R$ is $\wedge$L, $\pi$ ends in, e.g.,

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \varphi, \Gamma \Rightarrow \Delta\end{array}}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\,\wedge\text{L}$$

By the inductive hypothesis, there is a derivation $\delta_1$ of $x : \varphi, \Gamma' \Rightarrow \chi$ or $\Gamma' \Rightarrow \chi$ where $\Gamma'$ corresponds to $\Gamma$. In the latter case, we can take $\delta_1$ to be $\delta$. In the former case, observe that since $x : \varphi$ occurs in the end-sequent, $x : \varphi$ is not active in any inference with discharge label $x$ in $\delta_1$. Replace every axiom $x : \varphi \Rightarrow \varphi$ with the derivation $\delta_2$

$$\dfrac{x : \varphi \wedge \psi \Rightarrow \varphi \wedge \psi}{x : \varphi \wedge \psi \Rightarrow \varphi}\,\wedge\text{Elim}$$

and every occurrence of $x : \varphi$ by $x : \varphi \wedge \psi$, i.e., consider $\delta_1[\delta_2/x : \varphi]$. This is a derivation of $x : \varphi \wedge \psi, \Gamma' \Rightarrow \chi$.

8. If $R$ is Cut, $\pi$ ends in

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1 & \vdots\,\pi_2\\ \Gamma_1 \Rightarrow \varphi & \varphi, \Gamma_2 \Rightarrow \Delta\end{array}}{\Gamma_1, \Gamma_2 \Rightarrow \Delta}\,\text{Cut}$$

The inductive hypothesis yields proof $\delta_1$ of $\Gamma_1' \Rightarrow \varphi$ and $\delta_2$ of $x : \varphi, \Gamma_2' \Rightarrow \chi$ or of $\Gamma_2' \Rightarrow \chi$. In the latter case, take $\delta$ to be $\delta_2$. Otherwise, replace every axiom $x : \varphi \Rightarrow \varphi$ in $\delta_2$ by $\delta_1$ to obtain $\delta$ as the derivation $\delta_2[\delta_1/x : \varphi]$ of $\Gamma_1', \Gamma_2' \Rightarrow \chi$. □

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## Bibliography