nat.1 Natural Deduction with Sequents: N2c and N2i

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A derivation δ in **G1c** or **G1i** shows that its end-formula φ follows from its open assumptions. If Γ is the set of formulas ψ, where ψ^x occurs as an open assumptions in δ, we can write this as $\delta \vdash \Gamma \Rightarrow \varphi$. This suggests that we can formulate a natural deduction system also for sequents. In such a system, each sequent carries the information of which open assumptions the formula on the right depends on, i.e., which are open in the sub-derivation ending in a sequent. This is natural deduction with sequents, or natural deduction "in sequent style," and we call the resulting systems **N2c** and **N2i**.

To make the correspondence between **N1** and **N2** closer, let's take the sets Γ on the left to consist of labelled formulas $x : \psi$. Since rules in **N1** only operate on the formulas in the premises, i.e., the end-formulas of the immediate sub-derivations, rules of **N2** only operate on the succedent of the sequent. We can thus refer to the labelled formulas on the left of sequents simply as the context.

Instead of assumptions, **N2** proofs have as topmost sequents axioms of the form

$$x : \varphi \Rightarrow \varphi.$$

The rules correspond exactly to the rules of **N1** and are given in Table 1.

Because we must allow (as we do in **N1c** and **N1i**) that the rules $\to$Intro, $\neg$Intro, $\vee$Elim, and $\exists$Elim may, but need not actually discharge assumptions, we allow correct applications of the corresponding rules in **N2c** and **N2i** also when the corresponding context formulas φ, ψ, and $\varphi(a)$ are not present in the context of the relevant premise.

Proposition nat.1. *If δ is a derivation in* **N1c** *or* **N1i** *of φ, then there is a derivation δ' of $\Gamma \Rightarrow \varphi$ in* **N2c** *(***N2i***), where Γ is the set of all $x : \psi$ such that ψ^x is an open assumption of δ.*

Proof. By induction on $n = \mathrm{ht}(\delta)$. If $n = 0$, δ is a single open assumption φ^x. Then $\Gamma = \{x : \varphi\}$ and $\Gamma \Rightarrow \varphi$ is an axiom of **N2c** or **N2i**.

If $n > 0$ we distinguish cases according to the last inference in δ. We discuss just a one case; the rest are left as exercises.

Suppose the last inference is $\to$Intro. Then δ ends in

$$\dfrac{\begin{array}{c}[\psi]^x \\ \vdots\,\delta_1 \\ \chi\end{array}}{\psi \to \chi}\ {\scriptstyle x}\ \to\text{Intro}$$

By induction hypothesis there is a derivation δ_1' in **N2c** (**N2i**) of $\Gamma' \Rightarrow \chi$, where Γ' are the labelled assumptions open in δ_1. If ψ^x is an open assumption of δ_1, then it is discharged at the $\to$Intro inference, and the open assumptions

$$\frac{x : \varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\ {\to}\text{Intro} \qquad \frac{\Gamma_1 \Rightarrow \varphi \to \psi \qquad \Gamma_2 \Rightarrow \varphi}{\Gamma_1, \Gamma_2 \Rightarrow \psi}\ {\to}\text{Elim}$$

$$\frac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \varphi}\ {\wedge}\text{Elim}_1$$

$$\frac{\Gamma_1 \Rightarrow \varphi \qquad \Gamma_2 \Rightarrow \psi}{\Gamma_1, \Gamma_2 \Rightarrow \varphi \wedge \psi}\ {\wedge}\text{Intro} \qquad \frac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \psi}\ {\wedge}\text{Elim}_2$$

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi}\ {\vee}\text{Intro}_1 \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi}\ {\vee}\text{Intro}_2$$

$$\frac{\Gamma_1 \Rightarrow \varphi \vee \psi \qquad x : \varphi, \Gamma_2 \Rightarrow \chi \qquad y : \psi, \Gamma_3 \Rightarrow \chi}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \chi}\ {\vee}\text{Elim}$$

$$\frac{x : \varphi, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow \neg\varphi}\ {\neg}\text{Intro} \qquad \frac{\Gamma_1 \Rightarrow \neg\varphi \qquad \Gamma_2 \Rightarrow \varphi}{\Gamma_1, \Gamma_2 \Rightarrow \bot}\ {\neg}\text{Elim}$$

$$\frac{\Gamma \Rightarrow \varphi(a)}{\Gamma \Rightarrow \forall x\, \varphi(x)}\ \forall\text{Intro} \qquad \frac{\Gamma \Rightarrow \forall x\, \varphi(x)}{\Gamma \Rightarrow \varphi(t)}\ \forall\text{Elim}$$

$$\frac{\Gamma \Rightarrow \varphi(t)}{\Gamma \Rightarrow \exists x\, \varphi(x)}\ \exists\text{Intro} \qquad \frac{\Gamma_1 \Rightarrow \exists x\, \varphi(x) \qquad x : \varphi(a), \Gamma_2 \Rightarrow \chi}{\Gamma_1, \Gamma_2 \Rightarrow \chi}\ \exists\text{Elim}$$

$$\frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow \varphi}\ \bot_I \qquad \frac{x : \neg\varphi, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow \varphi}\ \bot_C$$

Table 1: Rules of **N2c** and **N2i**. In ∀Intro and ∃Elim, the constant symbol a must not occur in the conclusion sequent. Γ_i are sets of labelled formulas $x : \varphi$. The rules ∀Intro and ∃Elim must satisfy the eigenvariable condition: a must not occur in the conclusion sequent. **N2i** does not include $\bot_C$.

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of δ are $\Gamma = \Gamma' \setminus \{x : \psi\}$. In other words, δ_1' is a derivation of $x : \psi, \Gamma \Rightarrow \chi$, and we can take δ' to be

$$\frac{\begin{array}{c}\vdots \\ \delta_1' \\ \vdots \end{array}}{\begin{array}{r} x : B, \Gamma \Rightarrow \chi \\ \hline \Gamma \Rightarrow \psi \to \chi \end{array}\ {\to}\text{Intro}}$$

If ψ^x is not an open assumption of δ_1, then →Intro does not discharge an assumption, and the open assumptions of δ and δ_1 are the same. Since the context formula $x : \psi$ is not required to be present in the premise of →Intro

in **N2c** and **N2i**, we can take δ' to be

$$\dfrac{\begin{array}{c}\vdots\\ \delta_1'\\ \vdots\end{array}}{\Gamma \Rightarrow \chi}\ {\to}\text{Intro}$$

The cases where δ ends in another rule are similar. □

Proposition nat.2. *If δ is a derivation in* **N2c** *or* **N2i** *of $\Gamma \Rightarrow \varphi$, then there is a derivation δ' in* **N1c** *(***N1i***) with end-formula φ and open assumptions ψ^x for every $x : \psi \in \Gamma$.*

Proof. We again proceed by induction on the height n of δ. If $n = 0$, δ is an axiom, $x : \varphi \Rightarrow \varphi$. The derivation δ' is the assumption φ^x.

If $n > 0$, we distinguish cases according to the last inference in δ. For instance, if that inference is $\to$Intro, δ ends in

$$\dfrac{\begin{array}{c}\vdots\\ \delta_1\\ \vdots\end{array}}{x : \psi, \Gamma \Rightarrow \chi}\ {\to}\text{Intro} \qquad \text{or} \qquad \dfrac{\begin{array}{c}\vdots\\ \delta_1\\ \vdots\end{array}}{\Gamma \Rightarrow \chi}\ {\to}\text{Intro}$$

By inductive hypothesis, there is a derivation in **N1c** (**N1i**) of χ. In the first case, ψ^x is an open assumption of δ'; in the second case it is not. We can take the derivation δ' to be

$$\dfrac{\begin{array}{c}[\psi]^x\\ \vdots\\ \delta_1\\ \vdots\\ \chi\end{array}}{\psi \to \chi}\ {\to}\text{Intro}^{\,x} \qquad \text{or} \qquad \dfrac{\begin{array}{c}\vdots\\ \delta_1\\ \vdots\\ \chi\end{array}}{\psi \to \chi}\ {\to}\text{Intro}$$

respectively. □

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Bibliography