nat.1 Rules and Derivations

pt:nat:rp:sec

We begin by giving some definitions and fixing some terminology.

Consider the two rules in **N1c** and **N1i** that mention $\varphi \to \psi$.

$$\dfrac{\begin{array}{c}[\varphi]^x \\ \vdots \\ \psi\end{array}}{\varphi \to \psi}\, x \to\text{Intro} \qquad \dfrac{\varphi \to \psi \qquad \varphi}{\psi} \to\text{Elim}$$

The →Elim rule is simple enough. The formulas above the line are the *premises*. The formula on the left is the *principal* formula $\varphi \to \psi$. All elimination rules have one such premise, always on the far left. It is called the *major* premise of the inference. If there are others, such as an this case, they are called *minor* premises. The premise(s) of introduction rules are also called major premise(s). The formula below the line is the *conclusion*.

The →Intro rule only has one premise, and here the conclusion is the principal formula $\varphi \to \psi$. All introduction rules are like this: the conclusion is the principal formula; its main operator is the one being "introduced."

The notation $[\varphi]^x$ means the following. In a derivation, the →Intro inference is applied to a formula ψ as premise. The sub-derivation ending in ψ has a number of initial formulas. These are called *assumptions*. A derivation of ψ from one or more assumptions of the form φ is a derivation that φ entails ψ. When we apply the →Intro rule, we conclude $\varphi \to \psi$, and the conclusion no longer depends on the assumption φ. To indicate this, the assumptions of the form φ in the derivation including the →Intro inference are *discharged* or *closed*. The rule states which assumptions can be discharged. In the case of →Intro with conclusion $\varphi \to \psi$, that's assumptions that have the form φ, i.e., agree with the antecedent of the conditional. It is sometimes important to indicate which assumptions are discharged at which rule, and this is accomplished with the discharge label x. Importantly, rules that discharge assumptions don't *have to* discharge *all* assumptions of the requisite form, or indeed any at all. For instance, the following is correct, even though the first →Intro does not discharge any assumptions:

$$\dfrac{\dfrac{\dfrac{[\varphi]^y}{\psi \to \varphi}\, x \to\text{Intro}}{\varphi \to (\psi \to \varphi)}\, y \to\text{Intro}}{}$$

When a rule does not discharge any assumptions, we just leave out the discharge label (as we did here).

In the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^x \qquad [\varphi]^y}{\varphi \wedge \varphi} \wedge\text{Intro}}{\varphi} \wedge\text{Elim}}{\varphi \to \varphi}\, x \to\text{Intro}}{\varphi \to (\varphi \to \varphi)}\, y \to\text{Intro}$$

the first $\to$Intro inference discharges the left assumption φ^x, the second the right assumption φ^y. That is, all assumptions labelled x are discharged by the $\to$Intro with label x, and all assumptions with label y by the one labelled y. For this to work, it is important that all assumptions with the same label are also the same formula.

The quantifier rules are slightly more complicated. E.g., the rules in **N1i** for $\exists$ are:

$$\frac{\varphi(t)}{\exists x\,\varphi(x)}\,\exists\text{Intro} \qquad {}_{x}\frac{\exists x\,\varphi(x) \qquad \begin{array}{c}[\varphi(c)]^x\\ \vdots\\ \psi\end{array}}{\psi}\,\exists\text{Elim}$$

In the $\exists$Intro rule, the premise is $\varphi(t)$ where t is a closed term, i.e., a term without variables. Such an inference is correct—that is, it matches the schematic rule $\exists$Intro—if there is some formula φ, some variable x, and some closed term t such that the conclusion is $\exists x\,\varphi$ and premise is $\varphi[t/x]$. The $\exists$Elim allows us to discharge assumptions of the form $\varphi(c)$, i.e., for every inference there is some constant symbol c and assumptions of the form $\varphi[c/x]$ can be discharged. The discharged assumptions of one inference all have to use the same c. This constant is called the *eigenvariable* of the $\exists$Elim inference. The inference is only correct if c does not occur in an open assumption other than the discharged $\varphi(c)$, does not occur in ψ, and does not occur in φ. This is called the *eigenvariable condition*.

Derivations in **N1c** and **N1i** are finite trees of sequents that are inductively generated from axioms using the rules of inference. Each leaf is an assumption, possibly labelled with a discharge label, and every other formula follows from the formulas immediately above it by one of the inference rules of the system. If an occurence of a formula as an assumption in the tree above an inference has not been discharged by an inference above it, it is called *open* (at the inference).

Definition nat.1. *Derivations* in **N1c** and **N1i** are finite trees of formulas inductively as follows: pt:nat:rp: defn:proof

1. Every formula labelled by a discharge label such as x or y, by itself is a derivation. In a derivation consisting of a single formula, the formula counts as an open assumption. pt:nat:rp: defn:prf:assumption

2. If R is a rule with one, two, or three premises φ_1, φ_2, φ_3, conclusion φ, and δ_1, δ_2, δ_3 are derivations ending in φ_1, φ_2, φ_3, respectively, the pt:nat:rp: defn:prf:one-prem

$$\frac{\begin{array}{c}\vdots\\ \delta_1\\ \vdots\\ \varphi_1\end{array}}{\varphi}\,R \qquad \frac{\begin{array}{c}\vdots\\ \delta_1\\ \vdots\\ \varphi_1\end{array} \quad \begin{array}{c}\vdots\\ \delta_2\\ \vdots\\ \varphi_2\end{array}}{\varphi}\,R \qquad \frac{\begin{array}{c}\vdots\\ \delta_1\\ \vdots\\ \varphi_1\end{array} \quad \begin{array}{c}\vdots\\ \delta_2\\ \vdots\\ \varphi_2\end{array} \quad \begin{array}{c}\vdots\\ \delta_3\\ \vdots\\ \varphi_2\end{array}}{\varphi}\,R$$

   is a derivation, respectively, provided the assumption labels in the derivations are compatible (no two different formulas have the same discharge label) and the inferences are correct applications of R.

3. Topmost formula occurrences in the new derivation count as *open assumptions* of the derivation if they are open assumptions of δ_1, δ_2, or δ_3, except those occurrences discharged by the rule. If the rule is labelled x (or $x\,y$), then for $\to$Intro and $\neg$Intro, that is all open assumptions in δ_1 labelled x; for $\exists$Elim, it is all open assumptions labelled x in δ_2; for $\vee$Elim, it is all open assumptions labelled x in δ_2 and all open assumptions labelled y in δ_3.

The derivations used in the inductive construction of a derivation are called its *sub-derivations*. The sub-derivations ending in the premises of an inference are called the *immediate sub-derivations* of the inference. The rules of **N2c** and **N2i** are given in Table 1

Definition nat.2. The *height* $\mathrm{ht}(\delta)$ of a derivation δ is inductively defined as follows.

1. If δ consists of an assumption, $\mathrm{ht}(\delta) = 0$.
2. If δ ends in an inference with one premise with immediate sub-derivation δ_1, then $\mathrm{ht}(\delta) = \mathrm{ht}(\delta_1) + 1$.
3. If δ ends in an inference with two premises with immediate sub-derivations δ_1 and δ_2, then $\mathrm{ht}(\delta) = \max(\mathrm{ht}(\delta_1), \mathrm{ht}(\delta_2)) + 1$.
4. If δ ends in $\vee$Elim with immediate sub-derivations δ_1, δ_2, and δ_3, then $\mathrm{ht}(\delta) = \max(\mathrm{ht}(\delta_1), \mathrm{ht}(\delta_2), \mathrm{ht}(\delta_3)) + 1$.

If a sequent S has a derivation of height $\leq n$, we write $\vdash_n S$.

Example nat.3. In **N1i**, any formula counts as a derivation from itself as an open assumption. So

$$\varphi \wedge \psi^x$$

is a derivation δ_1. Its height is 0. From δ_1 we can obtain two derivations δ_2 and δ_3 by applying one of the two version of $\wedge$Elim

$$\frac{\varphi \wedge \psi^x}{\psi}\,\wedge\mathrm{Elim}_2 \qquad \frac{\varphi \wedge \psi^x}{\varphi}\,\wedge\mathrm{Elim}_1$$

The immediate sub-derivations of the last inference in δ_2 and in δ_3 is the same: $\varphi \wedge \psi$. Both derivations have the occurrence of $\varphi \wedge \psi$ as an open assumption. We have $\mathrm{ht}(\delta_1) = \mathrm{ht}(\delta_2) = 1$.

The rule $\wedge$Intro requires two premises of the form ψ and φ to justify a conclusion of the form $\psi \wedge \varphi$. Thus, the following is a derivation δ_4:

$$\delta_2\Big\{\frac{\dfrac{\varphi \wedge \psi^x}{\psi}\,\wedge\mathrm{Elim}_2 \qquad \dfrac{\varphi \wedge \psi^x}{\varphi}\,\wedge\mathrm{Elim}_1}{\psi \wedge \varphi}\,\wedge\mathrm{Intro}\Big\}\delta_3$$

Its height $\mathrm{ht}(\delta_4) = \max(\mathrm{ht}(\delta_2), \mathrm{ht}(\delta_3)) + 1 = \max(1, 1) + 1 = 2$. It has two occurences of $\varphi \wedge \psi$ as assumptions, both are open.

$$\dfrac{\begin{array}{c}[\varphi]^x\\ \vdots\\ \psi\end{array}}{\varphi \to \psi}\ {\scriptstyle x}\ \to\text{Intro} \qquad \dfrac{\varphi \to \psi \qquad \varphi}{\psi}\ \to\text{Elim}$$

$$\dfrac{\varphi \qquad \psi}{\varphi \wedge \psi}\ \wedge\text{Intro} \qquad \dfrac{\varphi \wedge \psi}{\varphi}\ \wedge\text{Elim}_1 \qquad \dfrac{\varphi \wedge \psi}{\psi}\ \wedge\text{Elim}_2$$

$$\dfrac{\varphi}{\varphi \vee \psi}\ \vee\text{Intro}_1 \qquad \dfrac{\psi}{\varphi \vee \psi}\ \vee\text{Intro}_2 \qquad \dfrac{\varphi \vee \psi \qquad \begin{array}{c}[\varphi]^x\\ \vdots\\ \chi\end{array} \qquad \begin{array}{c}[\psi]^y\\ \vdots\\ \chi\end{array}}{\chi}\ {\scriptstyle x\,y}\ \vee\text{Elim}$$

$$\dfrac{\begin{array}{c}[\varphi]^x\\ \vdots\\ \bot\end{array}}{\neg\varphi}\ {\scriptstyle x}\ \neg\text{Intro} \qquad \dfrac{\neg\varphi \qquad \varphi}{\bot}\ \neg\text{Elim}$$

$$\dfrac{\varphi(a)}{\forall x\, \varphi(x)}\ \forall\text{Intro} \qquad \dfrac{\forall x\, \varphi(x)}{\varphi(t)}\ \forall\text{Elim}$$

$$\dfrac{\varphi(t)}{\exists x\, \varphi(x)}\ \exists\text{Intro} \qquad \dfrac{\exists x\, \varphi(x) \qquad \begin{array}{c}[\varphi(a)]^x\\ \vdots\\ \chi\end{array}}{\chi}\ {\scriptstyle x}\ \exists\text{Elim}$$

$$\dfrac{\bot}{\varphi}\ \bot_I \qquad \dfrac{\begin{array}{c}[\neg\varphi]^x\\ \vdots\\ \bot\end{array}}{\varphi}\ {\scriptstyle x}\ \bot_C$$

Table 1: Rules of **N1i** and **N1c**. In ∀Intro the constant symbol a must not occur in the conclusion or an open assumption. In ∃Elim, a must not occur in the premises or an open assumption. **N1i** does not include $\bot_C$.

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tab:N1

Finally we can apply $\to$Intro to obtain δ_5:

$$\dfrac{\delta_2\left\{\dfrac{\dfrac{[\varphi\wedge\psi]^x}{\psi}\wedge\text{Elim}_2 \qquad \dfrac{[\varphi\wedge\psi]^x}{\varphi}\wedge\text{Elim}_1\right\}\delta_3}{\psi\wedge\varphi}\wedge\text{Intro}}{(\varphi\wedge\psi)\to(\psi\wedge\varphi)}\,{\scriptstyle x}\,\to\text{Intro}$$

Its height is $\text{ht}(\delta_4)+1=3$. The $\to$Intro inference discharges all occurrences of assumptions of the form $\varphi\wedge\psi$ labelled x, i.e., both previously open assumptions of the immediate subproof. Since those are the only assumptions of δ_5, it has no open assumptions, and so is a derivation *of* its end-formula $(\varphi\wedge\psi)\to(\psi\wedge\varphi)$.

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Bibliography