nat.1 Regular Derivations and Substitution

pt:nat:qua:sec

The rules for the quantifiers introduce some complications owing to the "eigenvariable conditions" that apply to the ∃Elim and ∀Intro rules. Most of these complications can be dealt with by ensuring that the constant symbol c in these inferences is never reused. Let's introduce the following definition.

Definition nat.1. A natural deduction derivation δ is *regular* if

1. no eigenvariable a is used as the eigenvariable of more than one ∃Elim or ∀Intro inference, and
2. if a is the eigenvariable of an ∃Elim or ∀Intro inference, it only occurs in the immediate sub-derivation leading to the minor premise, or the premise, respectively.

pt:nat:qua:lem:subst-regular

Lemma nat.2. *If δ is regular, ends in χ, c is a constant symbol not used as an eigenvariable in δ, and t is a term not containing any eigenvariables of δ, then $\delta[t/c]$ resulting from δ by replacing every occurrence of c by t is a regular derivation. The end-formula of $\delta[t/c]$ is $\chi[t/c]$. If φ^x is an open assumption of δ then $\varphi[t/c]^x$ is an open assumption of $\delta[t/c]$.*

Proof. By induction on $\mathrm{ht}(\delta)$. If $\mathrm{ht}(\delta) = 0$, it consists simply of an assumption φ^x. Then $\delta[t/c]$ is $\varphi[t/c]^x$.

If $\mathrm{ht}(\delta) > 0$ we distinguish cases according to the last inference in δ. The cases for rules for propositional connectives are easy and left as exercises. We consider the cases where δ ends in a quantifier inference.

1. If the last inference is ∀Intro, δ is of the form

$$\dfrac{\begin{array}{c}\vdots\\ \delta'\\ \vdots\end{array}}{\dfrac{\varphi(a,c)}{\forall x\, \varphi(x,c)}\,\forall\text{Intro}}$$

   By induction hypothesis, $\delta'[t/c]$ is a regular derivation of $\varphi(a,t)$. Since a is an eigenvariable of δ, a does not occur in t. Therefore, the last inference in $\delta[t/c]$,

$$\dfrac{\begin{array}{c}\vdots\\ \delta'[t/c]\\ \vdots\end{array}}{\dfrac{\varphi(a,t)}{\forall x\, \varphi(x,t)}\,\forall\text{Intro}}$$

   is correct: since t does not contain a, neither the conclusion nor any open assumption contains a.

2. If the last inference is ∀Elim, δ is of the form

$$\dfrac{\begin{array}{c}\vdots\\ \delta'\\ \vdots\end{array}}{\dfrac{\forall x\,\varphi(x,c)}{\varphi(s(c),c)}\,\forall\text{Elim}}$$

By induction hypothesis, $\delta'[t/c]$ is a regular derivation of $\forall x\,\varphi(x,t)$. (The term $s(c)$ in the conclusion may, but need not, contain c.) The last inference in $\delta[t/c]$,

$$\dfrac{\begin{array}{c}\vdots\\ \delta'[t/c]\\ \vdots\end{array}}{\dfrac{\forall x\,\varphi(x,t)}{\varphi(s(t),t)}\,\forall\text{Elim}}$$

is correct, and $\varphi(s(t),t) \equiv \varphi(s(c),c)[t/c]$. The case for ∃Intro is similar.

3. If the last inference is ∃Elim, δ is of the form

$$\dfrac{\exists x\,\varphi(x,c)\quad\begin{array}{c}[\varphi(a,c)]^x\\ \vdots\\ \chi\end{array}}{\chi}\,x\,\exists\text{Elim}$$

By induction hypothesis, $\delta_1'[t/c]$ and $\delta_2'[t/c]$ are regular derivations of $\exists x\,\varphi(x,c)$ and of χ from the open assumption $\varphi(a,t)$. Since a is an eigenvariable of δ, a does not occur in t. Therefore, the last inference in $\delta[t/c]$,

$$\dfrac{\begin{array}{c}\vdots\\ \delta'[t/c]\\ \vdots\end{array}}{\dfrac{\varphi(a,t)}{\forall x\,\varphi(x,t)}\,\forall\text{Intro}}$$

is correct: $\varphi(x,t)[a/x] \equiv \varphi(a,c)[t/c]$, neither the conclusion χ nor any open assumption contains a. □

Proposition nat.3. *If δ is a derivation in* **N1c** *or* **N1i**, *then there is a regular derivation δ' with the same structure (in particular, the same height) which differs from δ only by replacing eigenvariables.* *pt:nat:qua: prop:regular*

Proof. For the purpose of the proof only, call an eigenvariable inference of δ *clean* if its eigenvariable is not the eigenvariable of another inference nor does it occur anywhere in δ except the immediate sub-derivation of the inference, and *dirty* otherwise. Let n be the number of dirty eigenvariable inferences in δ. We

show, by induction on n, that δ can be transformed into the required regular derivation δ'. If $n = 0$, all eigenvariable inferences in δ are clean, and so δ is regular, and there is nothing to prove. Otherwise, pick a highest eigenvariable inference, say $\forall$Intro. The sub-derivation δ_1 of δ ending in it is of the form

$$\dfrac{\begin{array}{c}\vdots\\ \delta_2\\ \vdots\\ \varphi(c)\end{array}}{\forall x\, \varphi(x)}\ \forall\text{Intro}$$

Note that as c is an eigenvariable, it does not occur in $\forall x\, \varphi(x)$ or an open assumption. The proof δ_2 is regular, since it contains no eigenvariable inferences at all. Let a be a constant symbol not occurring in δ. By Lemma nat.2, $\delta_2[a/c]$ is a regular proof of $\varphi(a)$. By the eigenvariable condition, no open assumption of δ_2 contains c, hence no open assumption of $\delta_2[a/c]$ contains a. Thus we can apply $\forall$Intro. If we replace δ_1 in δ by the proof δ_1',

$$\dfrac{\begin{array}{c}\vdots\\ \delta_2[a/c]\\ \vdots\\ \varphi(a)\end{array}}{\forall x\, \varphi(x)}\ \forall\text{Intro}$$

we have replaced a dirty eigenvariable inference by a clean one, and the only difference is the replacement of the eigenvariable c by a. The resulting proof has fewer than n dirty eigenvariable inferences, and the result follows from the inductive hypothesis. □

Problem nat.1. Describe the case in the proof of Proposition nat.3 where the topmost eigenvariable inference is $\exists$Elim.

We will not appeal to the proposition just proved explicitly, but we will assume that all derivations considered are regular—if they are not, we can always replace eigenvariables to make them regular.

Lemma nat.2 and Proposition nat.3 also hold for **N2c** and **N2i**. We leave the proofs as exercises.

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Bibliography