Chapter udf

Natural Deduction

nat.1 Introduction

pt:nat:int:sec

Natural deduction is a family of derivation systems in which each logical connective or quantifier has a pair of inferences, one that "introduces" the operator (an introduction rule) and one that "eliminates" the operator (an elimination rule). For instance, →Intro has formulas of the form $\varphi \to \psi$ as conclusion, and →Elim has formulas of the form $\varphi \to \psi$ as premises.

The first two versions of natural deduction were introduced in the 1930s by Gerhard Gentzen and Stanisław Jaśkowski. Gentzen's is the system proof theorists discuss mostly, and Jaśkowski gave rise to the systems most widely used in teaching. In both systems, one can start with *assumptions*—formulas that don't need to be derived, but that can be used in deriving other formulas. An entire derivation may depend on such assumptions, and since those are unproven, the derivation does not establish its conclusion (its *end-formula*), but only that its conclusion follows from the assumptions.

Some inference rules are such that their conclusions no longer depend on some assumptions: they "discharge" assumptions. For instance, the →Intro rule takes a derivation of the conclusion ψ from an assumption φ, and has $\varphi \to \psi$ as its conclusion. The derivation ending in $\varphi \to \psi$ no longer depends on φ; the assumption φ is discharged. In Gentzen's systems, derivations are trees of formulas, the assumptions are the topmost formulas, and rules like →Intro carry labels that indicate which assumptions are discharged. In Jaśkowski's systems, parts of the derivation that depend on an assumption are set off in some way, e.g, indented behind a vertical line or in a box, with the assumption φ discharged and the consequent ψ of the conditional the first and last line in the box, and the conclusion $\varphi \to \psi$ of →Intro outside the box.

The systems are "natural" because the inference rules mirror how we (or mathematicians, at least) reason naturally. To establish a hypothetical result (a conditional), we assume the antecedent and use it to prove the consequent. That's →Intro. When we know a conditional $\varphi \to \psi$ and its antecedent φ, we conclude ψ. That's →Elim. We use proof-by-cases to show that, assuming a

disjunction $\varphi \vee \psi$, some conclusion holds. That's $\vee$Elim. To prove a general claim $\forall x\, \varphi(x)$ we prove that $\varphi(c)$ holds, for an arbitrary c. That's $\forall$Intro. And so on.

For instance here's a simple derivation of $\varphi \to (\varphi \vee \psi)$ in Gentzen-style natural deduction:

$$\dfrac{\dfrac{[\varphi]^x}{\varphi \vee \psi}\,\vee\text{Intro}}{\varphi \to (\varphi \vee \psi)}\,{}^x\,{\to}\text{Intro}$$

It starts with the assumption φ, uses $\vee$Intro to infer $\varphi \vee \psi$, and then $\to$Intro to infer $\varphi \to (\varphi \vee \psi)$. The $\to$Intro inference discharges the assumption φ; that's indicated by the label x.

Proof theorists prefer Gentzen's original systems working with trees of formulas, although one variant is also important in proof theory. A derivation of φ from assumptions Γ shows that φ follows from Γ, i.e., $\Gamma \vDash \varphi$. The inference rules of natural deduction can naturally be read as telling us something about such entailments, e.g., $\to$Intro says that if $\Gamma + \varphi \vDash \psi$ then $\Gamma \vDash \varphi \to \psi$. It is only natural to formulate the rules directly so that they operate on both the assumptions and the conclusion that follow from them in the premises and conclusion of the inference rule, i.e., they operate on sequents $\Gamma \Rightarrow \varphi$. The simple proof above would look like this in such a system:

$$\dfrac{\dfrac{x : \varphi \;\Rightarrow\; \varphi}{x : \varphi \;\Rightarrow\; \varphi \vee \psi}\,\vee\text{Intro}}{\Rightarrow\; \varphi \to (\varphi \vee \psi)}\,{}^x\,{\to}\text{Intro}$$

As you see, the rules are the same: they work on the formula to the right of $\Rightarrow$, the formulas on the left just list the assumptions they depend on at each step.

The complete set of introduction/elimination rule pairs is not complete for classical logic by itself. We need to add two additional rules that have to do with $\bot$. The first is the "intuitionistic absurdity rule" $\bot_I$, or "explosion," which says that from $\bot$ we can infer anything. The second is the classical principle of indirect proof $\bot_C$ (or the "classical absurdity rule") which says that if we can derive $\bot$ from $\neg\varphi$, we have derived φ. If we leave out $\bot_C$, we get a system appropriate for intuitionsitic logic. If we leave out both, we have what's known as "minimal logic."

We will discuss Gentzen-style natural deduction for classical and intuitionsitic logic. These are the systems **N1c** and **N1i**. The systems **N2c** and **N2i** are the corresponding systems working on sequents $\Gamma \Rightarrow \varphi$ rather than single formulas φ.

nat.2 Rules and Derivations

We begin by giving some definitions and fixing some terminology. pt:nat:rp:sec

Consider the two rules in **N1c** and **N1i** that mention $\varphi \to \psi$.

$$x\ \dfrac{\begin{array}{c}[\varphi]^x \\ \vdots \\ \psi\end{array}}{\varphi \to \psi}\ {\to}\text{Intro} \qquad \dfrac{\varphi \to \psi \qquad \varphi}{\psi}\ {\to}\text{Elim}$$

The →Elim rule is simple enough. The formulas above the line are the *premises*. The formula on the left is the *principal* formula $\varphi \to \psi$. All elimination rules have one such premise, always on the far left. It is called the *major* premise of the inference. If there are others, such as an this case, they are called *minor* premises. The premise(s) of introduction rules are also called major premise(s). The formula below the line is the *conclusion*.

The →Intro rule only has one premise, and here the conclusion is the principal formula $\varphi \to \psi$. All introduction rules are like this: the conclusion is the principal formula; its main operator is the one being "introduced."

The notation $[\varphi]^x$ means the following. In a derivation, the →Intro inference is applied to a formula ψ as premise. The sub-derivation ending in ψ has a number of initial formulas. These are called *assumptions*. A derivation of ψ from one or more assumptions of the form φ is a derivation that φ entails ψ. When we apply the →Intro rule, we conclude $\varphi \to \psi$, and the conclusion no longer depends on the assumption φ. To indicate this, the assumptions of the form φ in the derivation including the →Intro inference are *discharged* or *closed*. The rule states which assumptions can be discharged. In the case of →Intro with conclusion $\varphi \to \psi$, that's assumptions that have the form φ, i.e., agree with the antecedent of the conditional. It is sometimes important to indicate which assumptions are discharged at which rule, and this is accomplished with the discharge label x. Importantly, rules that discharge assumptions don't *have to* discharge *all* assumptions of the requisite form, or indeed any at all. For instance, the following is correct, even though the first →Intro does not discharge any assumptions:

$$y\ \dfrac{x\ \dfrac{[\varphi]^y}{\psi \to \varphi}\ {\to}\text{Intro}}{\varphi \to (\psi \to \varphi)}\ {\to}\text{Intro}$$

When a rule does not discharge any assumptions, we just leave out the discharge label (as we did here).

In the derivation

$$y\ \dfrac{x\ \dfrac{\dfrac{[\varphi]^x \qquad [\varphi]^y}{\varphi \wedge \varphi}\ {\wedge}\text{Intro}}{\varphi}\ {\wedge}\text{Elim}}{\varphi \to \varphi}\ {\to}\text{Intro}}{\varphi \to (\varphi \to \varphi)}\ {\to}\text{Intro}$$

the first →Intro inference discharges the left assumption φ^x, the second the right assumption φ^y. That is, all assumptions labelled x are discharged by the

→Intro with label x, and all assumptions with label y by the one labelled y. For this to work, it is important that all assumptions with the same label are also the same formula.

The quantifier rules are slightly more complicated. E.g., the rules in **N1i** for ∃ are:

$$\frac{\varphi(t)}{\exists x\, \varphi(x)}\ \exists\text{Intro} \qquad {}_{x}\frac{\exists x\, \varphi(x) \qquad \begin{array}{c}[\varphi(c)]^x\\ \vdots\\ \psi\end{array}}{\psi}\ \exists\text{Elim}$$

In the ∃Intro rule, the premise is $\varphi(t)$ where t is a closed term, i.e., a term without variables. Such an inference is correct—that is, it matches the schematic rule ∃Intro—if there is some formula φ, some variable x, and some closed term t such that the conclusion is $\exists x\, \varphi$ and premise is $\varphi[t/x]$. The ∃Elim allows us to discharge assumptions of the form $\varphi(c)$, i.e., for every inference there is some constant symbol c and assumptions of the form $\varphi[c/x]$ can be discharged. The discharged assumptions of one inference all have to use the same c. This constant is called the *eigenvariable* of the ∃Elim inference. The inference is only correct if c does not occur in an open assumption other than the discharged $\varphi(c)$, does not occur in ψ, and does not occur in φ. This is called the *eigenvariable condition*.

Derivations in **N1c** and **N1i** are finite trees of sequents that are inductively generated from axioms using the rules of inference. Each leaf is an assumption, possibly labelled with a discharge label, and every other formula follows from the formulas immediately above it by one of the inference rules of the system. If an occurence of a formula as an assumption in the tree above an inference has not been discharged by an inference above it, it is called *open* (at the inference).

Definition nat.1. *Derivations* in **N1c** and **N1i** are finite trees of formulas inductively as follows: pt:nat:rp: defn:proof

1. Every formula labelled by a discharge label such as x or y, by itself is a derivation. In a derivation consisting of a single formula, the formula counts as an open assumption. pt:nat:rp: defn:prf:assumption

2. If R is a rule with one, two, or three premises φ_1, φ_2, φ_3, conclusion φ, and δ_1, δ_2, δ_3 are derivations ending in φ_1, φ_2, φ_3, respectively, the pt:nat:rp: defn:prf:one-prem

$$\frac{\begin{array}{c}\vdots\,\delta_1\\ \varphi_1\end{array}}{\varphi}\ R \qquad \frac{\begin{array}{c}\vdots\,\delta_1\\ \varphi_1\end{array} \quad \begin{array}{c}\vdots\,\delta_2\\ \varphi_2\end{array}}{\varphi}\ R \qquad \frac{\begin{array}{c}\vdots\,\delta_1\\ \varphi_1\end{array} \quad \begin{array}{c}\vdots\,\delta_2\\ \varphi_2\end{array} \quad \begin{array}{c}\vdots\,\delta_3\\ \varphi_2\end{array}}{\varphi}\ R$$

is a derivation, respectively, provided the assumption labels in the derivations are compatible (no two different formulas have the same discharge label) and the inferences are correct applications of R.

3. Topmost formula occurrences in the new derivation count as *open assumptions* of the derivation if they are open assumptions of δ_1, δ_2, or δ_3, except those occurrences discharged by the rule. If the rule is labelled x (or $x\,y$), then for $\to$Intro and $\neg$Intro, that is all open assumptions in δ_1 labelled x; for $\exists$Elim, it is all open assumptions labelled x in δ_2; for $\vee$Elim, it is all open assumptions labelled x in δ_2 and all open assumptions labelled y in δ_3.

The derivations used in the inductive construction of a derivation are called its *sub-derivations*. The sub-derivations ending in the premises of an inference are called the *immediate sub-derivations* of the inference. The rules of **N2c** and **N2i** are given in Table nat.1

Definition nat.2. The *height* $\mathrm{ht}(\delta)$ of a derivation δ is inductively defined as follows.

1. If δ consists of an assumption, $\mathrm{ht}(\delta) = 0$.
2. If δ ends in an inference with one premise with immediate sub-derivation δ_1, then $\mathrm{ht}(\delta) = \mathrm{ht}(\delta_1) + 1$.
3. If δ ends in an inference with two premises with immediate sub-derivations δ_1 and δ_2, then $\mathrm{ht}(\delta) = \max(\mathrm{ht}(\delta_1), \mathrm{ht}(\delta_2)) + 1$.
4. If δ ends in $\vee$Elim with immediate sub-derivations δ_1, δ_2, and δ_3, then $\mathrm{ht}(\delta) = \max(\mathrm{ht}(\delta_1), \mathrm{ht}(\delta_2), \mathrm{ht}(\delta_3)) + 1$.

If a sequent S has a derivation of height $\leq n$, we write $\vdash_n S$.

Example nat.3. In **N1i**, any formula counts as a derivation from itself as an open assumption. So

$$\varphi \wedge \psi^x$$

is a derivation δ_1. Its height is 0. From δ_1 we can obtain two derivations δ_2 and δ_3 by applying one of the two version of $\wedge$Elim

$$\frac{\varphi \wedge \psi^x}{\psi}\,\wedge\mathrm{Elim}_2 \qquad \frac{\varphi \wedge \psi^x}{\varphi}\,\wedge\mathrm{Elim}_1$$

The immediate sub-derivations of the last inference in δ_2 and in δ_3 is the same: $\varphi \wedge \psi$. Both derivations have the occurrence of $\varphi \wedge \psi$ as an open assumption. We have $\mathrm{ht}(\delta_1) = \mathrm{ht}(\delta_2) = 1$.

The rule $\wedge$Intro requires two premises of the form ψ and φ to justify a conclusion of the form $\psi \wedge \varphi$. Thus, the following is a derivation δ_4:

$$\delta_2\Big\{\frac{\dfrac{\varphi \wedge \psi^x}{\psi}\,\wedge\mathrm{Elim}_2 \qquad \dfrac{\varphi \wedge \psi^x}{\varphi}\,\wedge\mathrm{Elim}_1}{\psi \wedge \varphi}\,\wedge\mathrm{Intro}\Big\}\delta_3$$

Its height $\mathrm{ht}(\delta_4) = \max(\mathrm{ht}(\delta_2), \mathrm{ht}(\delta_3)) + 1 = \max(1, 1) + 1 = 2$. It has two occurences of $\varphi \wedge \psi$ as assumptions, both are open.

$$\dfrac{\begin{array}{c}[\varphi]^x\\ \vdots\\ \psi\end{array}}{\varphi \to \psi}\ {\scriptstyle x}\ \to\text{Intro} \qquad \dfrac{\varphi \to \psi \qquad \varphi}{\psi}\ \to\text{Elim}$$

$$\dfrac{\varphi \qquad \psi}{\varphi \wedge \psi}\ \wedge\text{Intro} \qquad \dfrac{\varphi \wedge \psi}{\varphi}\ \wedge\text{Elim}_1 \qquad \dfrac{\varphi \wedge \psi}{\psi}\ \wedge\text{Elim}_2$$

$$\dfrac{\varphi}{\varphi \vee \psi}\ \vee\text{Intro}_1 \qquad \dfrac{\psi}{\varphi \vee \psi}\ \vee\text{Intro}_2 \qquad {\scriptstyle x\,y}\ \dfrac{\varphi \vee \psi \qquad \begin{array}{c}[\varphi]^x\\ \vdots\\ \chi\end{array} \qquad \begin{array}{c}[\psi]^y\\ \vdots\\ \chi\end{array}}{\chi}\ \vee\text{Elim}$$

$$\dfrac{\begin{array}{c}[\varphi]^x\\ \vdots\\ \bot\end{array}}{\neg\varphi}\ {\scriptstyle x}\ \neg\text{Intro} \qquad \dfrac{\neg\varphi \qquad \varphi}{\bot}\ \neg\text{Elim}$$

$$\dfrac{\varphi(a)}{\forall x\, \varphi(x)}\ \forall\text{Intro} \qquad \dfrac{\forall x\, \varphi(x)}{\varphi(t)}\ \forall\text{Elim}$$

$$\dfrac{\varphi(t)}{\exists x\, \varphi(x)}\ \exists\text{Intro} \qquad {\scriptstyle x}\ \dfrac{\exists x\, \varphi(x) \qquad \begin{array}{c}[\varphi(a)]^x\\ \vdots\\ \chi\end{array}}{\chi}\ \exists\text{Elim}$$

$$\dfrac{\bot}{\varphi}\ \bot_I \qquad {\scriptstyle x}\ \dfrac{\begin{array}{c}[\neg\varphi]^x\\ \vdots\\ \bot\end{array}}{\varphi}\ \bot_C$$

Table nat.1: Rules of **N1i** and **N1c**. In ∀Intro the constant symbol a must not occur in the conclusion or an open assumption. In ∃Elim, a must not occur in the premises or an open assumption. **N1i** does not include $\bot_C$.

pt:nat:rp:
tab:N1

Finally we can apply $\to$Intro to obtain δ_5:

$$\dfrac{\dfrac{\delta_2\left\{\dfrac{[\varphi\wedge\psi]^x}{\psi}\wedge\text{Elim}_2 \quad \dfrac{[\varphi\wedge\psi]^x}{\varphi}\wedge\text{Elim}_1\right\}\delta_3}{\psi\wedge\varphi}\wedge\text{Intro}}{(\varphi\wedge\psi)\to(\psi\wedge\varphi)}\,{\scriptstyle x}\,\to\text{Intro}$$

Its height is $\text{ht}(\delta_4)+1=3$. The $\to$Intro inference discharges all occurrences of assumptions of the form $\varphi\wedge\psi$ labelled x, i.e., both previously open assumptions of the immediate subproof. Since those are the only assumptions of δ_5, it has no open assumptions, and so is a derivation *of* its end-formula $(\varphi\wedge\psi)\to(\psi\wedge\varphi)$.

nat.3 Natural Deduction with Sequents: N2c and N2i

pt:nat:seq: sec

A derivation δ in **G1c** or **G1i** shows that its end-formula φ follows from its open assumptions. If Γ is the set of formulas ψ, where ψ^x occurs as an open assumptions in δ, we can write this as $\delta\vdash\Gamma\Rightarrow\varphi$. This suggests that we can formulate a natural deduction system also for sequents. In such a system, each sequent carries the information of which open assumptions the formula on the right depends on, i.e., which are open in the sub-derivation ending in a sequent. This is natural deduction with sequents, or natural deduction "in sequent style," and we call the resulting systems **N2c** and **N2i**.

To make the correspondence between **N1** and **N2** closer, let's take the sets Γ on the left to consist of labelled formulas $x:\psi$. Since rules in **N1** only operate on the formulas in the premises, i.e., the end-formulas of the immediate sub-derivations, rules of **N2** only operate on the succedent of the sequent. We can thus refer to the labelled formulas on the left of sequents simply as the context.

Instead of assumptions, **N2** proofs have as topmost sequents axioms of the form

$$x:\varphi\Rightarrow\varphi.$$

The rules correspond exactly to the rules of **N1** and are given in Table nat.2.

Because we must allow (as we do in **N1c** and **N1i**) that the rules $\to$Intro, $\neg$Intro, $\vee$Elim, and $\exists$Elim may, but need not actually discharge assumptions, we allow correct applications of the corresponding rules in **N2c** and **N2i** also when the corresponding context formulas φ, ψ, and $\varphi(a)$ are not present in the context of the relevant premise.

Proposition nat.4. *If δ is a derivation in* **N1c** *or* **N1i** *of φ, then there is a derivation δ' of $\Gamma\Rightarrow\varphi$ in* **N2c** *(***N2i***), where Γ is the set of all $x:\psi$ such that ψ^x is an open assumption of δ.*

Proof. By induction on $n=\text{ht}(\delta)$. If $n=0$, δ is a single open assumption φ^x. Then $\Gamma=\{x:\varphi\}$ and $\Gamma\Rightarrow\varphi$ is an axiom of **N2c** or **N2i**.

If $n>0$ we distinguish cases according to the last inference in δ. We discuss just a one case; the rest are left as exercises.

$$\frac{x:\varphi,\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi\to\psi}\ {\to}\text{Intro} \qquad \frac{\Gamma_1 \Rightarrow \varphi\to\psi \qquad \Gamma_2 \Rightarrow \varphi}{\Gamma_1,\Gamma_2 \Rightarrow \psi}\ {\to}\text{Elim}$$

$$\frac{\Gamma \Rightarrow \varphi\wedge\psi}{\Gamma \Rightarrow \varphi}\ {\wedge}\text{Elim}_1$$

$$\frac{\Gamma_1 \Rightarrow \varphi \qquad \Gamma_2 \Rightarrow \psi}{\Gamma_1,\Gamma_2 \Rightarrow \varphi\wedge\psi}\ {\wedge}\text{Intro} \qquad \frac{\Gamma \Rightarrow \varphi\wedge\psi}{\Gamma \Rightarrow \psi}\ {\wedge}\text{Elim}_2$$

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi\vee\psi}\ {\vee}\text{Intro}_1 \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi\vee\psi}\ {\vee}\text{Intro}_2$$

$$\frac{\Gamma_1 \Rightarrow \varphi\vee\psi \qquad x:\varphi,\Gamma_2 \Rightarrow \chi \qquad y:\psi,\Gamma_3 \Rightarrow \chi}{\Gamma_1,\Gamma_2,\Gamma_3 \Rightarrow \chi}\ {\vee}\text{Elim}$$

$$\frac{x:\varphi,\Gamma \Rightarrow \bot}{\Gamma \Rightarrow \neg\varphi}\ \neg\text{Intro} \qquad \frac{\Gamma_1 \Rightarrow \neg\varphi \qquad \Gamma_2 \Rightarrow \varphi}{\Gamma_1,\Gamma_2 \Rightarrow \bot}\ \neg\text{Elim}$$

$$\frac{\Gamma \Rightarrow \varphi(a)}{\Gamma \Rightarrow \forall x\,\varphi(x)}\ \forall\text{Intro} \qquad \frac{\Gamma \Rightarrow \forall x\,\varphi(x)}{\Gamma \Rightarrow \varphi(t)}\ \forall\text{Elim}$$

$$\frac{\Gamma \Rightarrow \varphi(t)}{\Gamma \Rightarrow \exists x\,\varphi(x)}\ \exists\text{Intro} \qquad \frac{\Gamma_1 \Rightarrow \exists x\,\varphi(x) \qquad x:\varphi(a),\Gamma_2 \Rightarrow \chi}{\Gamma_1,\Gamma_2 \Rightarrow \chi}\ \exists\text{Elim}$$

$$\frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow \varphi}\ \bot_I \qquad \frac{x:\neg\varphi,\Gamma \Rightarrow \bot}{\Gamma \Rightarrow \varphi}\ \bot_C$$

Table nat.2: Rules of **N2c** and **N2i**. In ∀Intro and ∃Elim, the constant symbol a must not occur in the conclusion sequent. Γ_i are sets of labelled formulas $x:\varphi$. The rules ∀Intro and ∃Elim must satisfy the eigenvariable condition: a must not occur in the conclusion sequent. **N2i** does not include $\bot_C$.

pt:nat:seq:
tab:N2

Suppose the last inference is →Intro. Then δ ends in

$$\begin{array}{c}[\psi]^x\\ \vdots\ \delta_1\\ \vdots\\ \chi\end{array}$$
$$x\ \frac{\qquad}{\psi\to\chi}\ {\to}\text{Intro}$$

By induction hypothesis there is a derivation δ_1' in **N2c** (**N2i**) of $\Gamma' \Rightarrow \chi$, where Γ' are the labelled assumptions open in δ_1. If ψ^x is an open assumption of δ_1, then it is discharged at the →Intro inference, and the open assumptions of δ are $\Gamma = \Gamma' \setminus \{x:\psi\}$. In other words, δ_1' is a derivation of $x:\psi,\Gamma \Rightarrow \chi$,

and we can take δ' to be

$$\dfrac{\begin{array}{c}\vdots\\ \delta_1'\\ \vdots\end{array}}{\begin{array}{r} x : B, \Gamma \Rightarrow \chi \\ \hline \Gamma \Rightarrow \psi \to \chi \end{array}} \to\text{Intro}$$

If ψ^x is not an open assumption of δ_1, then $\to$Intro does not discharge an assumption, and the open assumptions of δ and δ_1 are the same. Since the context formula $x : \psi$ is not required to be present in the premise of $\to$Intro in **N2c** and **N2i**, we can take δ' to be

$$\dfrac{\begin{array}{c}\vdots\\ \delta_1'\\ \vdots\end{array}}{\begin{array}{r} \Gamma \Rightarrow \chi \\ \hline \Gamma \Rightarrow \psi \to \chi \end{array}} \to\text{Intro}$$

The cases where δ ends in another rule are similar. □

Proposition nat.5. *If δ is a derivation in* **N2c** *or* **N2i** *of $\Gamma \Rightarrow \varphi$, then there is a derivation δ' in* **N1c** *(***N1i***) with end-formula φ and open assumptions ψ^x for every $x : \psi \in \Gamma$.*

Proof. We again proceed by induction on the height n of δ. If $n = 0$, δ is an axiom, $x : \varphi \Rightarrow \varphi$. The derivation δ' is the assumption φ^x.

If $n > 0$, we distinguish cases according to the last inference in δ. For instance, if that inference is $\to$Intro, δ ends in

$$\dfrac{\begin{array}{c}\vdots\\ \delta_1\\ \vdots\end{array}}{\begin{array}{r} x : \psi, \Gamma \Rightarrow \chi \\ \hline \Gamma \Rightarrow \psi \to \chi \end{array}} \to\text{Intro} \quad \text{or} \quad \dfrac{\begin{array}{c}\vdots\\ \delta_1\\ \vdots\end{array}}{\begin{array}{r} \Gamma \Rightarrow \chi \\ \hline \Gamma \Rightarrow \psi \to \chi \end{array}} \to\text{Intro}$$

By inductive hypothesis, there is a derivation in **N1c** (**N1i**) of χ. In the first case, ψ^x is an open assumption of δ'; in the second case it is not. We can take the derivation δ' to be

$$\dfrac{\begin{array}{c}[\psi]^x\\ \vdots\\ \delta_1\\ \vdots\\ \chi\end{array}}{\psi \to \chi}\, x \to\text{Intro} \quad \text{or} \quad \dfrac{\begin{array}{c}\vdots\\ \delta_1\\ \vdots\\ \chi\end{array}}{\psi \to \chi} \to\text{Intro}$$

respectively. □

nat.4 Regular Derivations and Substitution

The rules for the quantifiers introduce some complications owing to the "eigenvariable conditions" that apply to the ∃Elim and ∀Intro rules. Most of these complications can be dealt with by ensuring that the constant symbol c in these inferences is never reused. Let's introduce the following definition. pt:nat:qua:sec

Definition nat.6. A natural deduction derivation δ is *regular* if

1. no eigenvariable a is used as the eigenvariable of more than one ∃Elim or ∀Intro inference, and

2. if a is the eigenvariable of an ∃Elim or ∀Intro inference, it only occurs in the immediate sub-derivation leading to the minor premise, or the premise, respectively.

Lemma nat.7. *If δ is regular, ends in χ, c is a constant symbol not used as an eigenvariable in δ, and t is a term not containing any eigenvariables of δ, then $\delta[t/c]$ resulting from δ by replacing every occurrence of c by t is a regular derivation. The end-formula of $\delta[t/c]$ is $\chi[t/c]$. If φ^x is an open assumption of δ then $\varphi[t/c]^x$ is an open assumption of $\delta[t/c]$.* pt:nat:qua:lem:subst-regular

Proof. By induction on $\mathrm{ht}(\delta)$. If $\mathrm{ht}(\delta) = 0$, it consists simply of an assumption φ^x. Then $\delta[t/c]$ is $\varphi[t/c]^x$.

If $\mathrm{ht}(\delta) > 0$ we distinguish cases according to the last inference in δ. The cases for rules for propositional connectives are easy and left as exercises. We consider the cases where δ ends in a quantifier inference.

1. If the last inference is ∀Intro, δ is of the form

$$\dfrac{\begin{array}{c}\vdots\\ \delta'\\ \vdots\end{array}}{\dfrac{\varphi(a,c)}{\forall x\, \varphi(x,c)}\,\forall\text{Intro}}$$

   By induction hypothesis, $\delta'[t/c]$ is a regular derivation of $\varphi(a,t)$. Since a is an eigenvariable of δ, a does not occur in t. Therefore, the last inference in $\delta[t/c]$,

$$\dfrac{\begin{array}{c}\vdots\\ \delta'[t/c]\\ \vdots\end{array}}{\dfrac{\varphi(a,t)}{\forall x\, \varphi(x,t)}\,\forall\text{Intro}}$$

   is correct: since t does not contain a, neither the conclusion nor any open assumption contains a.

2. If the last inference is $\forall$Elim, δ is of the form

$$\dfrac{\begin{array}{c}\vdots\,\delta'\\ \forall x\,\varphi(x,c)\end{array}}{\varphi(s(c),c)}\ \forall\text{Elim}$$

By induction hypothesis, $\delta'[t/c]$ is a regular derivation of $\forall x\,\varphi(x,t)$. (The term $s(c)$ in the conclusion may, but need not, contain c.) The last inference in $\delta[t/c]$,

$$\dfrac{\begin{array}{c}\vdots\,\delta'[t/c]\\ \forall x\,\varphi(x,t)\end{array}}{\varphi(s(t),t)}\ \forall\text{Elim}$$

is correct, and $\varphi(s(t),t) \equiv \varphi(s(c),c)[t/c]$. The case for $\exists$Intro is similar.

3. If the last inference is $\exists$Elim, δ is of the form

$$x\ \dfrac{\begin{array}{c}\vdots\,\delta_1'\\ \exists x\,\varphi(x,c)\end{array}\qquad\begin{array}{c}[\varphi(a,c)]^x\\ \vdots\,\delta_2'\\ \chi\end{array}}{\chi}\ \exists\text{Elim}$$

By induction hypothesis, $\delta_1'[t/c]$ and $\delta_2'[t/c]$ are regular derivations of $\exists x\,\varphi(x,c)$ and of χ from the open assumption $\varphi(a,t)$. Since a is an eigenvariable of δ, a does not occur in t. Therefore, the last inference in $\delta[t/c]$,

$$\dfrac{\begin{array}{c}\vdots\,\delta'[t/c]\\ \varphi(a,t)\end{array}}{\forall x\,\varphi(x,t)}\ \forall\text{Intro}$$

is correct: $\varphi(x,t)[a/x] \equiv \varphi(a,c)[t/c]$, neither the conclusion χ nor any open assumption contains a. □

pt:nat:qua: prop:regular **Proposition nat.8.** *If δ is a derivation in* **N1c** *or* **N1i***, then there is a regular derivation δ' with the same structure (in particular, the same height) which differs from δ only by replacing eigenvariables.*

Proof. For the purpose of the proof only, call an eigenvariable inference of δ *clean* if its eigenvariable is not the eigenvariable of another inference nor does it occur anywhere in δ except the immediate sub-derivation of the inference, and *dirty* otherwise. Let n be the number of dirty eigenvariable inferences in δ. We

show, by induction on n, that δ can be transformed into the required regular derivation δ'. If $n = 0$, all eigenvariable inferences in δ are clean, and so δ is regular, and there is nothing to prove. Otherwise, pick a highest eigenvariable inference, say $\forall$Intro. The sub-derivation δ_1 of δ ending in it is of the form

$$\dfrac{\begin{array}{c}\vdots\\ \delta_2\\ \vdots\\ \varphi(c)\end{array}}{\forall x\, \varphi(x)}\ \forall\text{Intro}$$

Note that as c is an eigenvariable, it does not occur in $\forall x\, \varphi(x)$ or an open assumption. The proof δ_2 is regular, since it contains no eigenvariable inferences at all. Let a be a constant symbol not occurring in δ. By Lemma nat.7, $\delta_2[a/c]$ is a regular proof of $\varphi(a)$. By the eigenvariable condition, no open assumption of δ_2 contains c, hence no open assumption of $\delta_2[a/c]$ contains a. Thus we can apply $\forall$Intro. If we replace δ_1 in δ by the proof δ_1',

$$\dfrac{\begin{array}{c}\vdots\\ \delta_2[a/c]\\ \vdots\\ \varphi(a)\end{array}}{\forall x\, \varphi(x)}\ \forall\text{Intro}$$

we have replaced a dirty eigenvariable inference by a clean one, and the only difference is the replacement of the eigenvariable c by a. The resulting proof has fewer than n dirty eigenvariable inferences, and the result follows from the inductive hypothesis. □

Problem nat.1. Describe the case in the proof of Proposition nat.8 where the topmost eigenvariable inference is $\exists$Elim.

We will not appeal to the proposition just proved explicitly, but we will assume that all derivations considered are regular—if they are not, we can always replace eigenvariables to make them regular.

Lemma nat.7 and Proposition nat.8 also hold for **N2c** and **N2i**. We leave the proofs as exercises.

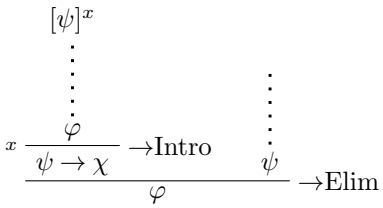
nat.5 Grafting Derivations

pt:nat:gra:sec

Derivations in **N1c** and **N1i** are derivations of formulas φ from assumptions ψ. Suppose ψ itself has a derivation. How do we combine the derivation of ψ with the derivation of φ from ψ to get a derivation of φ that doesn't use the assumption ψ?

One option is to remove the assumption ψ from the derivation of φ by applying $\to$Intro. Then we can combine the resulting derivation of $\psi \to \varphi$ with

the derivation of ψ to obtain



This is straightforward, but somewhat ugly: why introduce $\to$ only to eliminate it right away? It should be possible to avoid this kind of "detour" through $\psi \to \varphi$. (That they can always be avoided is in fact the content of the normalization theorem.)

In **N1c** and **N1i** we can actually avoid it in this case quite easily: we can simply replace the assumptions ψ^x by the entire derivation of ψ. This "grafting" of derivations onto the assumptions of others is quite useful and so we record it as a definition.

Definition nat.9. If δ_1 is a **N1c**-derivation (or **N1i**-derivation) of φ from open assumption ψ^x, and δ_2 is a **N1c**-derivation (**N1i**-derivation) of ψ, then $\delta_1[\delta_2/\psi^x]$ is the result of replacing every undischarged occurrence of ψ^x in δ_1 by δ_2.

Provided that δ_1 and δ_2 don't have different assumption formulas labelled with the same label, and no open assumption of δ_2 contains an eigenvariable of δ_1, the resulting $\delta_1[\delta_2/\psi^x]$ is a correct derivation, and its open assumptions are those of δ_1 together with those of δ_2. When grafting derivations, these conditions will typically be satisfied. The first is satisfied whenever δ_1 and δ_2 are subproofs of a larger derivation. If the second is not satisfied, we can rename the eigenvariables of δ_1 so that it is.

A corresponding definition applies to **N2c** and **N2i**.

Definition nat.10. If δ_1 is a **N2c**-derivation (or **N2i**-derivation) of $x : \psi, \Gamma_1 \Rightarrow \varphi$, and δ_2 is a **N1c**-derivation (**N1i**-derivation) of $\Gamma_2 \Rightarrow \psi$, then $\delta_1[\delta_2/x : \psi]$ is the result of replacing every axiom $x : \psi \Rightarrow \psi$ in δ_1 by δ_2, and adding Γ_2 to the context of every sequent below such axioms.

Proposition nat.11. *If $\delta_1 \vdash x : \psi, \Gamma_1 \Rightarrow \varphi$ and $\delta_2 \vdash \Gamma_2 \Rightarrow \psi$, then $\delta_1[\delta_2/x : \psi] \vdash \Gamma_1, \Gamma_2 \Rightarrow \varphi$, provided no eigenvariable of δ_1 occurs in Γ_2.*

Proof. By induction on $\mathrm{ht}(\delta)$. □

nat.6 Translating from G2i to N2i

pt:nat:tgi:sec

The antecedents of sequents in **G2i** are multisets Γ of formulas, while those of sequents in **N2i** are sets Γ' of labelled formulas with the property that every label occurs only once. Let's say a set Γ' of labelled formulas *corresponds to*

the multiset Γ if, for every formula φ, the number of elements of Γ' of the form $x : \varphi$ is less than or equal to the multiplicity of φ (i.e., the number of copies of φ) in Γ.

Proposition nat.12. *If* **G2i** $+$ Cut $\vdash \Gamma \Rightarrow \Delta$ *then* **N2i** $\vdash \Gamma' \Rightarrow \chi$, *where* $\chi \equiv \bot$ *if* Δ *is empty and* $\Delta = \{\varphi\}$ *otherwise, and* Γ' *corresponds to* Γ. *pt:nat:tgi: prop:G2i-to-N2i*

Proof. We show that if π is a derivation of $\Gamma \Rightarrow \Delta$ in **G2ci**, then there is a Γ' corresponding to Γ and a derivation δ of $\Gamma' \Rightarrow \chi$ in **N2i**. We do this by induction on $n = \mathrm{ht}(\pi)$.

If $n = 0$, then π is an axiom. If it is the axiom $\bot \Rightarrow$, δ is the axiom $x : \bot \Rightarrow \bot$, i.e., $\Gamma' = \{x : \bot\}$ and $\chi \equiv \bot$. If π is an axiom of the form $\varphi \Rightarrow \varphi$, then δ is $x : \varphi \Rightarrow \varphi$.

If $n > 0$, π ends in a rule R. The inductive hypothesis guarantees that there are derivations of sequents corresponding to the premises of that rule in **N2c**. We can assume that all discharge labels in those proofs are distinct, and that a label x only occurs above an inference that discharges it, if it is discharged at all (by relabelling if necessary). We then show how these derivations can be combined to provide proofs of a suitable $\Gamma' \Rightarrow \chi$. We distinguish cases according to R.

1. If R is WR, π ends in

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\\ \Gamma \Rightarrow\end{array}}{\Gamma \Rightarrow \varphi}\ \mathrm{WR}$$

   The induction hypothesis applied to π_1 yields a derivation of $\Gamma' \Rightarrow \bot$. We apply $\bot_I$ to get a derivation of $\Gamma' \Rightarrow \varphi$.

2. If R is WL, π ends in

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\\ \Gamma \Rightarrow \Delta\end{array}}{\varphi, \Gamma \Rightarrow \Delta}\ \mathrm{WR}$$

   The induction hypothesis applied to π_1 yields a derivation of $\Gamma' \Rightarrow \chi$. Which we take as δ. Note that if Γ' corresponds to Γ, it also corresponds to φ, Γ as the number of $x : \varphi$ in Γ' is no greater than the number of copies of φ in Γ, and so also no greater than the number of copies of φ in $\Gamma \cup \{\varphi\}$.

3. If R is CL, π ends in

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\\ \varphi, \varphi, \Gamma \Rightarrow \Delta\end{array}}{\varphi, \Gamma \Rightarrow \Delta}\ \mathrm{CL}$$

The induction hypothesis applied to π_1 yields a derivation of $\Pi, \Gamma' \Rightarrow \chi$, where $\Pi \subseteq \{x : \varphi, y : \varphi\}$. If $\Pi = \{x : \varphi, y : \varphi\}$, replace all labelled formulas $y : \varphi$ in π_1 by $x : \varphi$. Since x occurs in the context of the end-sequent of π_1 we can assume that no inference in π_1 discharges a formula of the form $x : \psi$, i.e., no $x : \psi$ on the left side of a sequent in π_1 is active. Consequently, the result is still a derivation in **N2i**.

4. If R is $\to$R, π ends in

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\\ \varphi, \Gamma \Rightarrow \psi\end{array}}{\Gamma \Rightarrow \varphi \to \psi} \to\text{R}$$

   By the inductive hypothesis, there is a derivation δ_1 of $x : \varphi, \Gamma' \Rightarrow \psi$ or $\Gamma' \Rightarrow \psi$ where Γ' corresponds to Γ. We can take δ_1 to be δ plus an application of $\to$Intro.

5. If R is $\to$L, π ends in

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\\ \Gamma_1 \Rightarrow \varphi\end{array} \qquad \begin{array}{c}\vdots\\ \pi_2\\ \vdots\\ \psi, \Gamma_2 \Rightarrow \Delta\end{array}}{\varphi \to \psi, \Gamma_1, \Gamma_2 \Rightarrow \Delta} \to\text{L}$$

   The inductive hypothesis yields derivations δ_1 of $\Gamma_1' \Rightarrow \varphi$ and δ_2 of $x : \psi, \Gamma_2' \Rightarrow \chi$ or of of $\Gamma_2' \Rightarrow \chi$. If δ_2 is of the latter form, δ_2 can serve as δ. Otherwise, we can relabel all formulas and discharge labels in π_1 to ensure that no label occurs in both δ_1 and δ_2. Then $\Gamma_1' \cup \Gamma_2'$ corresponds to $\Gamma_1 \cup \Gamma_2$. Since $x : \psi$ occurs in the end-sequent of δ_2, $x : \psi$ cannot be active in any inference with discharge label x in δ_2. Replace every axiom $x : \psi \Rightarrow \psi$ in δ_2 with the derivation δ_3

$$\dfrac{x : \varphi \to \psi \Rightarrow \varphi \to \psi \qquad \begin{array}{c}\vdots\\ \delta_1\\ \vdots\\ \Gamma_1' \Rightarrow \varphi\end{array}}{x : \varphi \to \psi \Rightarrow \psi} \to\text{Elim}$$

   and every occurrence of $x : \psi$ in δ_2 by $x : \varphi \to \psi$. The result $\delta_2[\delta_3/x : \psi]$ is a derivation of $x : \varphi \to \psi, \Gamma_1', \Gamma_2' \Rightarrow \chi$.

6. If R is $\wedge$R, π ends in

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\\ \Gamma_1 \Rightarrow \varphi\end{array} \qquad \begin{array}{c}\vdots\\ \pi_2\\ \vdots\\ \Gamma_2 \Rightarrow \psi\end{array}}{\Gamma_1, \Gamma_2 \Rightarrow \varphi \wedge \psi} \wedge\text{R}$$

The inductive hypothesis yields proof δ_1 of $\Gamma_1' \Rightarrow \varphi$ and δ_2 of $\Gamma_2' \Rightarrow \varphi$. By relabelling all formulas and discharge labels in π_2 we can ensure that no label occurs in both δ_1 and δ_2. Then $\Gamma_1' \cup \Gamma_2'$ corresponds to $\Gamma_1 \cup \Gamma_2$. We obtain a derivation δ by applying $\wedge$Intro to the end-sequents of δ_1 and δ_2.

7. If R is $\wedge$L, π ends in, e.g.,

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\end{array}}{\begin{array}{r}\varphi, \Gamma \Rightarrow \Delta\\ \hline \varphi \wedge \psi, \Gamma \Rightarrow \Delta\end{array}} \wedge\mathrm{L}$$

   By the inductive hypothesis, there is a derivation δ_1 of $x : \varphi, \Gamma' \Rightarrow \chi$ or $\Gamma' \Rightarrow \chi$ where Γ' corresponds to Γ. In the latter case, we can take δ_1 to be δ. In the former case, observe that since $x : \varphi$ occurs in the end-sequent, $x : \varphi$ is not active in any inference with discharge label x in δ_1. Replace every axiom $x : \varphi \Rightarrow \varphi$ with the derivation δ_2

$$\dfrac{x : \varphi \wedge \psi \Rightarrow \varphi \wedge \psi}{x : \varphi \wedge \psi \Rightarrow \varphi} \wedge\mathrm{Elim}$$

   and every occurrence of $x : \varphi$ by $x : \varphi \wedge \psi$, i.e., consider $\delta_1[\delta_2/x : \varphi]$. This is a derivation of $x : \varphi \wedge \psi, \Gamma' \Rightarrow \chi$.

8. If R is Cut, π ends in

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\end{array}}{\Gamma_1 \Rightarrow \varphi} \qquad \dfrac{\begin{array}{c}\vdots\\ \pi_2\\ \vdots\end{array}}{\varphi, \Gamma_2 \Rightarrow \Delta} \;\mathrm{Cut}$$
$$\Gamma_1, \Gamma_2 \Rightarrow \Delta$$

   The inductive hypothesis yields proof δ_1 of $\Gamma_1' \Rightarrow \varphi$ and δ_2 of $x : \varphi, \Gamma_2' \Rightarrow \chi$ or of $\Gamma_2' \Rightarrow \chi$. In the latter case, take δ to be δ_2. Otherwise, replace every axiom $x : \varphi \Rightarrow \varphi$ in δ_2 by δ_1 to obtain δ as the derivation $\delta_2[\delta_1/x : \varphi]$ of $\Gamma_1', \Gamma_2' \Rightarrow \chi$. □

nat.7 Translating from N2 to G2

pt:nat:tni: sec
pt:nat:tni: prop:N2-to-G2

Proposition nat.13. *If* **N2c** (**N2i**) $\vdash \Gamma \Rightarrow \varphi$ *then* **G2c** (**N2i**) + Cut $\vdash \Gamma' \Rightarrow \Delta$ *where* Γ' *is the multiset of formulas resulting from* Γ *by removing labels, and* $\Delta = \{\varphi\}$ *if* φ *is not* $\bot$*, and* $\Delta = \emptyset$ *if it is.*

Proof. The proof is again by induction on the height of the derivation δ of $\Gamma \Rightarrow \varphi$. If δ is an axiom $x : \varphi \Rightarrow \varphi$, π is the corresponding axiom $\varphi \Rightarrow \varphi$ or $\bot \Rightarrow$ if $\varphi \equiv \bot$.

If δ ends in an inference rule, we distinguish cases:

1. If R is $\to$Intro with principal formula $\psi \to \chi$ and $x : \psi$ is present in the context of the premise but not the conclusion, then δ ends in

$$\dfrac{\begin{array}{c}\vdots\\ \delta_1\\ \vdots\\ x : \psi, \Gamma \Rightarrow \chi\end{array}}{\Gamma \Rightarrow \psi \to \chi}\ \to\text{Intro}$$

   The inductive hypothesis yields a derivation π_1 of $\psi, \Gamma' \Rightarrow \chi$ (or of $\psi, \Gamma' \Rightarrow$ if $\chi \equiv \bot$), from which we obtain π by applying $\to$R (after applying WR in case $\chi \equiv \bot$).

2. If R is $\to$Intro, but $x : \psi$ is not present in the context of the premise, then δ ends in

$$\dfrac{\begin{array}{c}\vdots\\ \delta_1\\ \vdots\\ \Gamma \Rightarrow \chi\end{array}}{\Gamma \Rightarrow \psi \to \chi}\ \to\text{Intro}$$

   The inductive hypothesis yields a derivation π_1 of $\Gamma' \Rightarrow \chi$, from which we obtain π by applying WL with ψ, and then $\to$R.

3. If R is $\to$Elim, δ ends in

$$\dfrac{\begin{array}{c}\vdots\\ \delta_1\\ \vdots\\ \Gamma_1 \Rightarrow \psi \to \varphi\end{array} \qquad \begin{array}{c}\vdots\\ \delta_2\\ \vdots\\ \Gamma_2 \Rightarrow \psi\end{array}}{\Gamma_1, \Gamma_2 \Rightarrow \varphi}\ \to\text{Elim}$$

   where $\Gamma = \Gamma_1 \cup \Gamma_2$. The inductive hypothesis yields derivations π_1 of $\Gamma_1' \Rightarrow \psi \to \varphi$ and π_2 of $\Gamma_2' \Rightarrow \psi$. We obtain the derivation π as

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\\ \Gamma_1' \Rightarrow \psi \to \varphi\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ \pi_2\\ \vdots\\ \Gamma_2' \Rightarrow \psi\end{array} \qquad \varphi \Rightarrow \varphi}{\psi \to \varphi, \Gamma_2' \Rightarrow \varphi}\ \to\text{L}}{\Gamma_1', \Gamma_2' \Rightarrow \varphi}\ \text{Cut}$$

   The multiset $\Gamma_1' \cup \Gamma_2'$ may not equal $\Gamma' = (\Gamma_1 \cup \Gamma_2)'$: e.g., if both Γ_1 and Γ_2 contain $x : \chi$ then $\Gamma_1' \cup \Gamma_2'$ contains two copies of χ but $(\Gamma_1 \cup \Gamma_2)'$ contains only one copy. We can correct this by adding suitable CL inferences.

   In the event that $\varphi \equiv \bot$, the inductive hypothesis applied to δ_2 yields a derivation of $\Gamma_2' \Rightarrow$, from which we can obtain the derivation π by adding suitable WL inferences and a WR with φ.

4. If R is $\bot_C$, δ ends in

$$\dfrac{\begin{array}{c}\vdots\,\delta_1\\ x:\neg\varphi, \Gamma \Rightarrow \bot\end{array}}{\Gamma \Rightarrow \varphi}\,\bot_C$$

By inductive hypothesis, there is a derivation π_1 of $\neg\varphi, \Gamma' \Rightarrow$. We take the derivation π to be

$$\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\Rightarrow \varphi, \neg\varphi}\,\neg\mathrm{R} \qquad \begin{array}{c}\vdots\,\pi_1\\ x:\neg\varphi, \Gamma' \Rightarrow\end{array}}{\Gamma' \Rightarrow \varphi}\,\mathrm{Cut}$$

Note that only the translation of $\bot_C$ results in a derivation π with more than one formula on the right. That is, the translation of a **N2i** derivation is a **G2i**-derivation. □

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Bibliography