nat.1 Introduction

pt:nat:int:sec

Natural deduction is a family of derivation systems in which each logical connective or quantifier has a pair of inferences, one that "introduces" the operator (an introduction rule) and one that "eliminates" the operator (an elimination rule). For instance, $\to$Intro has formulas of the form $\varphi \to \psi$ as conclusion, and $\to$Elim has formulas of the form $\varphi \to \psi$ as premises.

The first two versions of natural deduction were introduced in the 1930s by Gerhard Gentzen and Stanisław Jaśkowski. Gentzen's is the system proof theorists discuss mostly, and Jaśkowski gave rise to the systems most widely used in teaching. In both systems, one can start with *assumptions*—formulas that don't need to be derived, but that can be used in deriving other formulas. An entire derivation may depend on such assumptions, and since those are unproven, the derivation does not establish its conclusion (its *end-formula*), but only that its conclusion follows from the assumptions.

Some inference rules are such that their conclusions no longer depend on some assumptions: they "discharge" assumptions. For instance, the $\to$Intro rule takes a derivation of the conclusion ψ from an assumption φ, and has $\varphi \to \psi$ as its conclusion. The derivation ending in $\varphi \to \psi$ no longer depends on φ; the assumption φ is discharged. In Gentzen's systems, derivations are trees of formulas, the assumptions are the topmost formulas, and rules like $\to$Intro carry labels that indicate which assumptions are discharged. In Jaśkowski's systems, parts of the derivation that depend on an assumption are set off in some way, e.g, indented behind a vertical line or in a box, with the assumption φ discharged and the consequent ψ of the conditional the first and last line in the box, and the conclusion $\varphi \to \psi$ of $\to$Intro outside the box.

The systems are "natural" because the inference rules mirror how we (or mathematicians, at least) reason naturally. To establish a hypothetical result (a conditional), we assume the antecedent and use it to prove the consequent. That's $\to$Intro. When we know a conditional $\varphi \to \psi$ and its antecedent φ, we conclude ψ. That's $\to$Elim. We use proof-by-cases to show that, assuming a disjunction $\varphi \vee \psi$, some conclusion holds. That's $\vee$Elim. To prove a general claim $\forall x\, \varphi(x)$ we prove that $\varphi(c)$ holds, for an arbitrary c. That's $\forall$Intro. And so on.

For instance here's a simple derivation of $\varphi \to (\varphi \vee \psi)$ in Gentzen-style natural deduction:

$$\dfrac{\dfrac{[\varphi]^x}{\varphi \vee \psi}\,\vee\text{Intro}}{\varphi \to (\varphi \vee \psi)}\,{}^{x}\,\to\text{Intro}$$

It starts with the assumption φ, uses $\vee$Intro to infer $\varphi \vee \psi$, and then $\to$Intro to infer $\varphi \to (\varphi \vee \psi)$. The $\to$Intro inference discharges the assumption φ; that's indicated by the label x.

Proof theorists prefer Gentzen's original systems working with trees of formulas, although one variant is also important in proof theory. A derivation of φ from assumptions Γ shows that φ follows from Γ, i.e., $\Gamma \vDash \varphi$. The inference rules of natural deduction can naturally be read as telling us something

about such entailments, e.g., →Intro says that if $\Gamma + \varphi \vDash \psi$ then $\Gamma \vDash \varphi \to \psi$. It is only natural to formulate the rules directly so that they operate on both the assumptions and the conclusion that follow from them in the premises and conclusion of the inference rule, i.e., they operate on sequents $\Gamma \Rightarrow \varphi$. The simple proof above would look like this in such a system:

$$\cfrac{\cfrac{\cfrac{}{x : \varphi \Rightarrow \varphi}}{x : \varphi \Rightarrow \varphi \vee \psi}\vee\text{Intro}}{\Rightarrow \varphi \to (\varphi \vee \psi)}\to\text{Intro}\,x$$

As you see, the rules are the same: they work on the formula to the right of ⇒, the formulas on the left just list the assumptions they depend on at each step.

The complete set of introduction/elimination rule pairs is not complete for classical logic by itself. We need to add two additional rules that have to do with ⊥. The first is the "intuitionistic absurdity rule" $\bot_I$, or "explosion," which says that from ⊥ we can infer anything. The second is the classical principle of indirect proof $\bot_C$ (or the "classical absurdity rule") which says that if we can derive ⊥ from $\neg\varphi$, we have derived φ. If we leave out $\bot_C$, we get a system appropriate for intuitionsitic logic. If we leave out both, we have what's known as "minimal logic."

We will discuss Gentzen-style natural deduction for classical and intuitionsitic logic. These are the systems **N1c** and **N1i**. The systems **N2c** and **N2i** are the corresponding systems working on sequents $\Gamma \Rightarrow \varphi$ rather than single formulas φ.

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Bibliography