nat.1 Grafting Derivations

pt:nat:gra:sec

Derivations in **N1c** and **N1i** are derivations of formulas φ from assumptions ψ. Suppose ψ itself has a derivation. How do we combine the derivation of ψ with the derivation of φ from ψ to get a derivation of φ that doesn't use the assumption ψ?

One option is to remove the assumption ψ from the derivation of φ by applying $\to$Intro. Then we can combine the resulting derivation of $\psi \to \varphi$ with the derivation of ψ to obtain

$$\dfrac{\dfrac{\begin{array}{c}[\psi]^x\\ \vdots\\ \varphi\end{array}}{\psi \to \chi}\,x \to\text{Intro} \qquad \begin{array}{c}\vdots\\ \psi\end{array}}{\varphi} \to\text{Elim}$$

This is straightforward, but somewhat ugly: why introduce $\to$ only to eliminate it right away? It should be possible to avoid this kind of "detour" through $\psi \to \varphi$. (That they can always be avoided is in fact the content of the normalization theorem.)

In **N1c** and **N1i** we can actually avoid it in this case quite easily: we can simply replace the assumptions ψ^x by the entire derivation of ψ. This "grafting" of derivations onto the assumptions of others is quite useful and so we record it as a definition.

Definition nat.1. If δ_1 is a **N1c**-derivation (or **N1i**-derivation) of φ from open assumption ψ^x, and δ_2 is a **N1c**-derivation (**N1i**-derivation) of ψ, then $\delta_1[\delta_2/\psi^x]$ is the result of replacing every undischarged occurrence of ψ^x in δ_1 by δ_2.

Provided that δ_1 and δ_2 don't have different assumption formulas labelled with the same label, and no open assumption of δ_2 contains an eigenvariable of δ_1, the resulting $\delta_1[\delta_2/\psi^x]$ is a correct derivation, and its open assumptions are those of δ_1 together with those of δ_2. When grafting derivations, these conditions will typically be satisfied. The first is satisfied whenever δ_1 and δ_2 are subproofs of a larger derivation. If the second is not satisfied, we can rename the eigenvariables of δ_1 so that it is.

A corresponding definition applies to **N2c** and **N2i**.

Definition nat.2. If δ_1 is a **N2c**-derivation (or **N2i**-derivation) of $x : \psi, \Gamma_1 \Rightarrow \varphi$, and δ_2 is a **N1c**-derivation (**N1i**-derivation) of $\Gamma_2 \Rightarrow \psi$, then $\delta_1[\delta_2/x : \psi]$ is the result of replacing every axiom $x : \psi \Rightarrow \psi$ in δ_1 by δ_2, and adding Γ_2 to the context of every sequent below such axioms.

Proposition nat.3. *If $\delta_1 \vdash x : \psi, \Gamma_1 \Rightarrow \varphi$ and $\delta_2 \vdash \Gamma_2 \Rightarrow \psi$, then $\delta_1[\delta_2/x : \psi] \vdash \Gamma_1, \Gamma_2 \Rightarrow \varphi$, provided no eigenvariable of δ_1 occurs in Γ_2.*

Proof. By induction on $\mathrm{ht}(\delta)$. □

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Bibliography