cut.1 The Midsequent Theorem and Herbrand's Theorem

pt:cut:mid: sec

An important consequence of the cut-elimination theorem is the midsequent theorem. It states that if there is a derivation π of $\Gamma \Rightarrow \Delta$ in which every formula is prenex, then there is a derivation π' containing a sequent $S \equiv \Pi \Rightarrow \Lambda$ (called the *midsequent*) such that all inferences above S in π' are propositional and all inferences below S are quantifier inferences. (A formula φ is *prenex* (or in *prenex form*) if it is of the form

$$\mathsf{Q}_1 x_1 \dots \mathsf{Q}_n x_n \, \psi(x_1, \dots, x_n)$$

where each Q_i is either $\forall$ or $\exists$.) An important consequence of the midsequent theorem is Herbrand's theorem. Its general case is a little hard to describe, but very generally it takes the form: for every derivable sentence φ of first-order logic, there is a propositional tautology φ' from which φ can be derived. Here is the version for sentences of the form $\exists x \, \psi(x)$ where ψ is quantifier-free:

Theorem cut.1 (Herbrand's Theorem). *Suppose that $\psi(x)$ is quantifier-free and that $\vdash \exists x \, \psi(x)$. Then there are terms $t_1, \dots, t_n$ such that $\vdash \psi(t_1) \vee \dots \vee \psi(t_n)$.*

The formula $\psi(t_1) \vee \dots \vee \psi(t_n)$ is called a *Herbrand disjunction* of $\exists x \, \psi(x)$. Since ψ is quantifier-free, this disjunction is also quantifier-free. So if it is derivable, it is derivable by a cut-free proof and consequently with a derivation that only uses propositional inferences. Hence, it is a tautology.

The difficult part of Herbrand's Theorem is showing that a Herbrand disjunction exists for every derivable formula $\exists x \, \psi(x)$. The converse, i.e., that if $\exists x \, \psi(x)$ has a Herbrand disjunction, then it is provable, is easy. In fact, the Herbrand disjunction entails $\exists x \, \psi(x)$. E.g., here is a derivation in **G3c** for the case where $n = 2$:

$$\dfrac{\dfrac{\dfrac{\psi(t_1) \Rightarrow \exists x \, \psi(x), \psi(t_1), \psi(t_2) \qquad \psi(t_2) \Rightarrow \exists x \, \psi(x), \psi(t_1), \psi(t_2)}{\psi(t_1) \vee \psi(t_2) \Rightarrow \exists x \, \psi(x), \psi(t_1), \psi(t_2)} \vee\mathrm{L}}{\psi(t_1) \vee \psi(t_2) \Rightarrow \exists x \, \psi(x), \psi(t_2)} \exists\mathrm{R}}{\psi(t_1) \vee \psi(t_2) \Rightarrow \exists x \, \psi(x)} \exists\mathrm{R}$$

To extract the Herbrand disjunction from a derivation π of $\Rightarrow \exists x \, \psi(x)$, consider the case where all $\exists$R inferences in a cut-free derivation π are at the end:

$$\begin{array}{l} \vdots \\ \vdots \, \pi_1 \\ \vdots \\ \Rightarrow \exists x \, \psi(x), \psi(t_1), \dots, \psi(t_n) \\ \hline \Rightarrow \exists x \, \psi(x), \psi(t_2), \dots, \psi(t_n) \quad \exists\mathrm{R} \\ \vdots \\ \vdots \, (n-1) \times \exists\mathrm{R} \\ \vdots \\ \Rightarrow \exists x \, \psi(x) \end{array}$$

If these are all the $\exists$R inferences in π with $\exists x\, \psi(x)$ active, then there are no other quantifier inferences in π_1'. That's because $\psi(x)$ contains no other quantifiers, and the end-sequent doesn't contain any formulas on the left. In other words, $\Rightarrow \exists x\, \psi(x), \psi(t_1), \ldots, \psi(t_n)$ is the mid-sequent of π. Moreover, since there are no quantifier inferences above the mid-sequent, all sequents above it contain $\exists x\, \psi(x)$ and these are neither active nor principal. So they can be deleted from the derivation π_1, and we get a derivation of $\Rightarrow \psi(t_1), \ldots, \psi(t_n)$, and using $n - 1$ applications of $\vee$R, a derivation of the Herbrand disjunction $\Rightarrow \psi(t_1) \vee \cdots \vee \psi(t_n)$.

The trouble is that we can't assume that even a cut-free derivation of $\Rightarrow \exists x\, \psi(x)$ has all its $\exists$R inferences in one block at the end. There might be inferences in between these $\exists$R inferences in which some $\psi(t_i)$ is principal. Hence, the need for the midsequent theorem.

Theorem cut.2 (Midsequent Theorem). *If there is a* **G3c**-*derivation* π *of* $\Gamma \Rightarrow \Delta$ *where all formulas in* Γ *and* Δ *are prenex, then there is a derivation* π' *containing a sequent* $\Pi \Rightarrow \Lambda$ *such that all inferences below it are quantifier inferences and all inferences above it are propositional inferences.* *pt:cut:mid: thm:midsequent*

Proof. Define the *order* of a quantifier inference in π as the number of propositional inferences below it, and the order $o(\pi)$ of π as the sum of the orders of all its quantifier inferences. If $o(\pi) = 0$, then no quantifier inference has a propositional inference below it, and we are done: Since all quantifier inferences have one premise, there is a single topmost quantifier inference. Its premise is the midsequent.

If $o(\pi) > 0$, pick a bottom-most quantifier inference with order > 0. Since it has order > 0, there must be a propositional inference below it. Since it is a bottom-most one, its conclusion can't be the premise of a quantifier inference: that second, lower quantifier inference would also have a propositional inference below it otherwise. We distinguish (many!) cases according to what quantifier inference we have, and what the propositional inference below it is. Because the end-sequent consists of prenex formulas only, the principal formula of the quantifier inference can't be active in the following propositional inference. Otherwise the principal formula of that inference would have a quantifier in the scope of a propositional connective. By the sub-formula property, this would have to be a sub-formula of a formula in the end-sequent. So, the principal formula of the quantifier inference is an element of the context of the lower propositional inference.

Here are two examples:

First, suppose we have a $\forall$R which ends in the right premise of $\rightarrow$L. Then

the relevant sub-derivation of π ends in the sub-derivation π_1.

$$\dfrac{\dfrac{\vdots\,\pi_2}{\Gamma' \Rightarrow \Delta', \forall x\,\varphi(x), \psi} \qquad \dfrac{\dfrac{\vdots\,\pi_3}{\chi, \Gamma' \Rightarrow \Delta', \varphi(c)}}{\chi, \Gamma' \Rightarrow \Delta', \forall x\,\varphi(x)}\,\forall\mathrm{R}}{\psi \to \chi, \Gamma' \Rightarrow \Delta', \forall x\,\varphi(x)}\,{\to}\mathrm{L}$$

We can assume that π is regular, and so the eigenvariable c does not occur outside π_3; in particular, it does not occur in $\psi \to \chi$ or in π_2. Now consider the proof π_1':

$$\dfrac{\dfrac{\vdots\,\pi_2'}{\Gamma' \Rightarrow \Delta', \forall x\,\varphi(x), \varphi(c), \psi} \qquad \dfrac{\vdots\,\pi_3}{\chi, \Gamma' \Rightarrow \Delta', \forall x\,\varphi(x), \varphi(c)}}{\dfrac{\psi \to \chi, \Gamma' \Rightarrow \Delta', \forall x\,\varphi(x), \varphi(c)}{\psi \to \chi, \Gamma' \Rightarrow \Delta', \forall x\,\varphi(x), \forall x\,\varphi(x)}\,\forall\mathrm{R}}\,{\to}\mathrm{L}$$

where π_2' is obtained from π_2 by weakening with $\varphi(c)$ on the right. (This does not add any inferences, in particular, no quantifier inferences which might affect the order. It also does not violate any eigenvariable conditions in π_2, since no constants in $\varphi(c)$ are used as eigenvariables in π_2.) The eigenvariable condition on $\forall$R is still satisfied, since c does not occur in $\psi \to \chi$.

We invoke the admissibility of contraction to obtain a derivation π_1'' of $\psi \to \chi, \Gamma' \Rightarrow \Delta', \forall x\,\varphi(x)$. The proof of the admissibility of CR for $\forall x\,\varphi(x)$ and the proof of invertibility of $\forall$R used in that proof did not change the order of any inference: it, at most, deleted inferences. Thus, there are no more quantifier inferences in π_1'' than in π_1, and each quantifier inference in π_1'' has the same number of proposional inferences below it as its corresponding inference in π_1—with the exception of the last $\forall$R: it had a $\to$L below it in π, which we have moved above it in π_1''. Replace π_1 in π by π_1''. The resulting derivation has a lower order, since at least the order of the $\forall$R inference has been reduced by (at least) 1. Now apply the inductive hypothesis.

The cases where the quantifier inference is $\exists$R is even easier, since contraction is not needed and no eigenvariable condition needs to be worried about. E.g., suppose $\exists$R is followed by $\wedge$R (this time as the left premise). The relevant sub-derivation is π_1:

$$\dfrac{\dfrac{\dfrac{\vdots\,\pi_2}{\Gamma' \Rightarrow \Delta', \exists x\,\varphi(x), \varphi(t), \psi}}{\Gamma' \Rightarrow \Delta', \exists x\,\varphi(x), \psi}\,\exists\mathrm{R} \qquad \dfrac{\vdots\,\pi_3}{\Gamma' \Rightarrow \Delta', \exists x\,\varphi(x), \chi}}{\Gamma' \Rightarrow \Delta', \exists x\,\varphi(x), \psi \wedge \chi}\,{\wedge}\mathrm{R}$$

Replace π_1 in π by π'_1:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi_2\\ \vdots\end{array}}{\Gamma' \Rightarrow \Delta', \exists x\, \varphi(x), \varphi(t), \psi} \qquad \dfrac{\begin{array}{c}\vdots\\ \pi'_3\\ \vdots\end{array}}{\Gamma' \Rightarrow \Delta', \exists x\, \varphi(x), \varphi(t), \chi}\,\wedge\mathrm{R}}{\dfrac{\Gamma' \Rightarrow \Delta', \exists x\, \varphi(x), \varphi(t), \psi \wedge \chi}{\Gamma' \Rightarrow \Delta', \exists x\, \varphi(x), \psi \wedge \chi}\,\exists\mathrm{R}}$$

where π'_3 is obtained from π_3 by weakening with $\varphi(t)$ on the right. This weakening involves merely adding $\varphi(t)$ to the right side of every sequent in π_3, so does not change the order of any quantifier inference. The order of quantifier inferences is the same in π' as in π, except the order of the $\exists$R inference permuted with $\wedge$R has decreased by 1. The inductive hypothesis applies. □

Problem cut.1. Describe the cases in the proof of Theorem cut.2 where $\exists$L ends in the premise of $\to$R, and where $\forall$L ends in the left premise of $\vee$L.

Proof of Herbrand's Theorem. Suppose π ends in $\Rightarrow \exists x\, \varphi(x)$. By Theorem cut.2, we can transform π into a derivation π' with a midsequent S, which must be of the form

$$\Rightarrow \exists x\, \varphi(x), \varphi(t_1), \ldots, \varphi(t_n).$$

Since all inferences above S are propositional, $\exists x\, \varphi(x)$ cannot be principal in an inference above S. Since it is not atomic, it cannot be principal in an axiom. Since $\exists x\, \varphi(x)$ cannot be a proper subformula of $\varphi(t_1)$ (remember that $\varphi(x)$ is quantifier-free), it can also not be active above S. Deleting $\exists x\, \varphi(x)$ from the sub-derivation ending in S therefore yields a correct derivation of $\Rightarrow \varphi(t_1), \ldots, \varphi(t_n)$, from which a derivation of $\Rightarrow \varphi(t_1) \vee \cdots \vee \varphi(t_n)$ can be obtained by $n - 1$ $\vee$R inferences. □

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Bibliography