cut.1 Introduction

pt:cut:int: sec

The Cut rule is the rule

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\ \text{Cut}$$

(The formula φ is called the *cut formula*.)

Cut was included in Gentzen's original sequent calculus **LK**. Gentzen's *Hauptsatz* ("main theorem") states that every sequent derivable in **LK** is also derivable without using the Cut rule, i.e., that Cut can be "eliminated" from **LK**-proofs. Our systems **G1c** and **G3c** do not contain Cut, but the question can be raised for them as well: can Cut be eliminated from derivations that use Cut plus the other rules of **G1c** (or those of **G3c**)? In our established terminology it can be asked equivalently as: Is Cut an admissible rule of **G1c** (or **G3c**).

The cut elimination theorem is perhaps the most important and celebrated result of proof theory. It first of all establishes that **G1c** and **G3c** don't need a rule which you might, at first blush, think is clearly required. For when you try to formalize actual proofs, you'll notice that you need a way to deal with the use of lemmas. Mathematicians like to prove results, and then appeal to these results in proofs of other results. So you might first formalize a proof of result L (the lemma) as the derivation of a sequent $\Rightarrow L$. Then you formalize the proof of the second result R, the proof of which uses L, as the derivation of a sequent $L \Rightarrow R$. How do you know that there is a proof of R, i.e., a derivation of just $\Rightarrow R$? You need a way to combine the two derivations, and the Cut rule would allow you to do that.

We mentioned that **G1c** and **G3c** are complete, and so they prove any sequent you might expect to be able to prove—all valid sequents. This can be proved directly. But at the time Gentzen was writing, only axiomatic derivation systems were known to be complete. To show **LK** complete, Gentzen provided a translation of derivations in the axiomatic system into **LK**-derivations. This translation used Cut essentially. Cut elimination is not just an important result for classical logic: many other logics have sequent calculi. For some logics, a direct proof of completeness for their sequent calculi is hard or impossible, so translations from other systems (using Cut) are the only way to show completeness. Then proving cut elimination proof-theoretically is necessary to establish that Cut is superfluous and that the system without Cut is complete.

Cut elimination is also important because it can be applied to obtain other, independently interesting results. Two cases we'll consider are the Craig interpolation lemma and Herbrand' theorem.

The ideas used to prove cut elimination typically involve showing how a derivation that uses Cut can be transformed into one without. These transformations are usually "local," i.e., we take a part of a derivation (usually a sub-derivation ending in a Cut inference) and replace it with another sub-derivation of the same sequent but in which the Cut inference has been removed or replaced by others. Such arguments provide step-by-step algorithms

for transforming derivations with Cut into ones without. This provides more information when cut elimination procedures are used in applications. For instance, there are model theoretic proofs of the interpolation lemma the form "if $\varphi \to \psi$" is valid then an interpolant exists (never mind what and interpolant is for the moment). A proof-theoretic argument using cut elimination provides an algorithm that takes a derivation of $\varphi \to \psi$ and returns an interpolant. In contrast to the model theoretic argument, the proof theoretic argument is *constructive*.

There are two popular ways to prove the cut elimination theorem. One (going back to Gentzen) shows that you can remove top-most cuts in a proof, and then just removes top-most cuts until none are left. The other (originally due to Schütte and Tait) focuses on the most complex cuts in a derivation, and successively removes cuts that are maximal in the sense that no other cut has a cut formula of higher depth. They each have their advantages and drawbacks. For some systems, one approach works while another is hard to implement or even impossible. So, we describe both approaches. We'll prove cut elimination for **G3c**. In fact, we'll show not that Cut can be eliminated, but *context-sharing cut* $\mathrm{Cut}_{\mathrm{CS}}$:

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}$$

If a sequent calculus allows weakening and contraction (either as actual rules like in **G1c** or as admissible rules like in **G3c**), Cut can simulate $\mathrm{Cut}_{\mathrm{CS}}$, and vice versa. We choose $\mathrm{Cut}_{\mathrm{CS}}$ over Cutbecause the first cut elimination strategy is easier to describe for $\mathrm{Cut}_{\mathrm{CS}}$, and the second strategy only works for $\mathrm{Cut}_{\mathrm{CS}}$.

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Bibliography