cut.1 Maehara's Lemma and Craig's Interpolation Theorem

pt:cut:itp: sec

The interpolation theorem, due to William Craig, states that if $\varphi \vDash \psi$ then there is a sentence χ (the *interpolant*) such that $\varphi \vDash \chi$ and $\chi \vDash \psi$. Importantly, the interpolant χ can be chosen so that the constant symbols and predicate symbols occurring in χ all occur in both φ and ψ. (We assume that no function symbols are involved.) Without this restriction, φ and ψ could trivially serve as interpolants.

The interpolation theorem holds for classical first-order logic and has been called the "last important property of first-order logic" to be discovered. It has important applications in model theory, and automated reasoning. Its original motivation and application was in the philosophy of science, since it can be used to study the primitives by which a theory can or cannot be axiomatized. It also implies Beth's definability theorem, which states that if a theory can implicitly define a concept, it can define it explicitly.

Like many results in mathematical logic, the interpolation theorem can be proved both using the methods of model theory and those of proof theory. The advantage of proof theoretic methods is that not only do they show the existence of an interpolant, but provide an algorithm that computes the interpolant from a derivation that $\varphi \vDash \psi$, e.g., a derivation in **G3c** of $\varphi \Rightarrow \psi$.

Let $\mathcal{L}(\varphi)$ be the set of constant symbols and predicate symbols in φ and $\mathcal{L}(\Gamma) = \bigcup\{\mathcal{L}(\varphi) : \varphi \in \Gamma\}$. Throughout this section, we assume we are dealing with formulas without function symbols.

pt:cut:itp: thm:interpolation

Theorem cut.1 (Craig's Interpolation Theorem). *Let the language of φ be $\mathcal{L}(\varphi)$ and that of ψ be $\mathcal{L}(\psi)$. If* **G3c** $\vdash \varphi \Rightarrow \psi$ *then there is a formula χ in the language $\mathcal{L}(\chi) \subseteq \mathcal{L}(\varphi) \cap \mathcal{L}(\psi)$ such that* **G3c** $\vdash \varphi \Rightarrow \chi$ *and* **G3c** $\vdash \chi \Rightarrow \psi$.

We prove a generalization of the interpolation theorem to make use of the structure of derivations in **G3c**. To this end, imagine the antecendent and succedents of sequents to be separated into two parts each. We write $\Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2$ to indicate that Γ_1 and Δ_1 belong to the first part and Γ_2 and Δ_2 to the second part. The interpolation theorem then follows immediately from the following lemma.

pt:cut:itp: lem:maehara

Lemma cut.2 (Maehara's Lemma). *If* **G3c** $\vdash \Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2$ *then there is a formula χ in the language $\mathcal{L}(\chi) \subseteq \mathcal{L}(\Gamma_1, \Delta_1) \cap \mathcal{L}(\Gamma_2, \Delta_2)$ such that* **G3c** $\vdash \Gamma_1 \Rightarrow \Delta_1, \chi$ *and* **G3c** $\vdash \chi, \Gamma_2 \Rightarrow \Delta_2$.

Proof. Assume that $\pi \vdash \Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2$. We prove the claim by induction on $n = \mathrm{ht}(\pi)$.

If $n = 0$, then $\Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2$ is an axiom, and some atomic formula φ occurs in both Γ_1, Γ_2 and Δ_1, Δ_2. There are four cases depending on whether φ belongs to the first part or the second on the left, and whether it belongs to the first or second part on the right.

1. $\varphi \in \Gamma_1$ and $\varphi \in \Delta_1$. We want a χ such that both $\Gamma_1 \Rightarrow \Delta_1, \chi$ and $\chi, \Gamma_2 \Rightarrow \Delta_2$ are derivable. The first is already an axiom regardless of how we pick χ since φ occurs on both the left and right. Our only option to guarantee that $\chi, \Gamma_2 \Rightarrow \Delta_2$ is derivable is to let $\chi \equiv \bot$.

2. $\varphi \in \Gamma_1$ and $\varphi \in \Delta_2$. Then $\Gamma_1 \Rightarrow \Delta_1, \varphi$ and $\varphi, \Gamma_2 \Rightarrow \Delta_2$ are both axioms, so we can pick $\chi \equiv \varphi$.

3. $\varphi \in \Gamma_2$ and $\varphi \in \Delta_1$. Then $\varphi, \Gamma_1 \Rightarrow \Delta_1$ and $\Gamma_2 \Rightarrow \Delta_2, \varphi$ would be axioms, but they have φ on the wrong sides. So φ isn't the interpolant. However, using $\neg$R and $\neg$L we can derive $\Gamma_1 \Rightarrow \Delta_1, \neg\varphi$ and $\neg\varphi, \Gamma_2 \Rightarrow \Delta_2$.

4. $\varphi \in \Gamma_2$ and $\varphi \in \Delta_2$. Exercise.

The cases where $\Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2$ is an axiom because $\bot \in \Gamma_1$ or $\bot \in \Gamma_2$ are left as exercises.

If $n > 0$, π ends in a logical inference. We have to distinguish cases cases according to which rule was used, and to which part the principal formula belongs. In each case we can use the induction hypothesis that the lemma holds for all derivations π_1 with $\mathrm{ht}(\pi_1) < n$, and however the end-sequent of π_1 may be partitioned into two parts. Let's consider some specific cases.

1. π ends in $\neg$R with principal formula $\neg\varphi$. We have two cases: depending on whether $\neg\varphi$ belongs to the first or second part, π ends in either

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \varphi, \Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2\end{array}}{\Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1, \neg\varphi \,;\, \Delta_2}\,\neg\mathrm{R}$$

   or

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \Gamma_1 \,;\, \varphi, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2\end{array}}{\Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2, \neg\varphi}\,\neg\mathrm{R}$$

   Note that if the principal formula $\neg\varphi$ belongs to the first part we assigned the active formula φ also to the first part the premise. This is a choice we get to make. The inductive hypothesis would also apply if we assigned the active formula to the other part, but then we would not be able to find an interpolant (exercise: try it).

   Consider the first situation first, where $\neg\varphi$ belongs to the first part. By inductive hypothesis, there is an interpolant of the end-sequent of π_1, i.e., a formula χ_1 such that both

$$\varphi, \Gamma_1 \Rightarrow \Delta_1, \chi_1 \qquad \text{and} \qquad \chi_1, \Gamma_2 \Rightarrow \Delta_2$$

are provable. What we need is a formula χ such that both of the following are derivable:

$$\Gamma_1 \Rightarrow \Delta_1, \neg\varphi, \chi \qquad \text{and} \qquad \chi, \Gamma_2 \Rightarrow \Delta_2$$

Clearly, we can just take $\chi \equiv \chi_1$: the inductive hypothesis already told us that the sequent on the right is derivable, and the sequent on the left follows from the sequent we got from the inductive hypothesis by $\neg$R. The inductive hypothesis also tells us that $\mathcal{L}(\chi_1) \subseteq \mathcal{L}(\varphi, \Gamma_1, \Delta_1) \cap \mathcal{L}(\Gamma_2, \Delta_2)$. Since $\mathcal{L}(\neg\varphi) = \mathcal{L}(\varphi)$ and $\mathcal{L}(\chi) = \mathcal{L}(\chi_1)$ we have that $\mathcal{L}(\chi_1) \subseteq \mathcal{L}(\Gamma_1, \Delta_1, \neg\varphi) \cap \mathcal{L}(\Gamma_2, \Delta_2)$.

In the second scenario the situation is slightly different in that $\neg\varphi$ belongs to the second part. We assign the active formula in the premise also to the second part and apply the inductive hypothesis to

$$\Gamma_1 \Rightarrow \Delta_1, \chi_1 \qquad \text{and} \qquad \chi_1, \varphi, \Gamma_2 \Rightarrow \Delta_2$$

are provable. If we again apply the $\neg$R rule to the second of these, we get derivations of

$$\Gamma_1 \Rightarrow \Delta_1, \chi_1 \qquad \text{and} \qquad \chi_1, \Gamma_2 \Rightarrow \Delta_2, \neg\varphi$$

These are again exactly the sequents we want to be derivable if χ_1 is an interpolant; we again let $\chi \equiv \chi_1$. $\mathcal{L}(\chi_1) \subseteq \mathcal{L}(\Gamma_1, \Delta_1) \cap \mathcal{L}(\Gamma_2, \Delta_2, \neg\varphi)$ as before.

2. π ends in $\to$R with principal formula $\varphi \to \psi$. We again have two cases, depending on whether $\varphi \to \psi$ belongs to the first or second part. The derivation π ends in either

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1 \\ \vdots \\ \varphi, \Gamma_1 \,;\, \Gamma_2 \;\Rightarrow\; \Delta_1, \psi \,;\, \Delta_2\end{array}}{\Gamma_1 \,;\, \Gamma_2 \;\Rightarrow\; \Delta_1, \varphi \to \psi \,;\, \Delta_2} \to\text{R}$$

or

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1 \\ \vdots \\ \Gamma_1 \,;\, \varphi, \Gamma_2 \;\Rightarrow\; \Delta_1 \,;\, \Delta_2, \psi\end{array}}{\Gamma_1 \,;\, \Gamma_2 \;\Rightarrow\; \Delta_1 \,;\, \Delta_2, \varphi \to \psi} \to\text{R}$$

Again we have assigned the active formulas to the same part in the premise as the principal formula in the conclusion. In the first secenario, we apply the inductive hypothesis and get derivations of

$$\varphi, \Gamma_1 \Rightarrow \Delta_1, \psi, \chi_1 \qquad \text{and} \qquad \chi_1, \Gamma_2 \Rightarrow \Delta_2$$

The sequent on the left is a suitable premise for →R. So we have the following derivable sequents:

$$\Gamma_1 \Rightarrow \Delta_1, \varphi \to \psi, \chi_1 \qquad \text{and} \qquad \chi_1, \Gamma_2 \Rightarrow \Delta_2$$

This means that χ_1 is also an interpolant for the end-sequent of π and we can take $\chi \equiv \chi_1$ once more.

The other scenario is handled the same way.

3. π ends in →L with principal formula $\varphi \to \psi$. If $\varphi \to \psi$ belongs to the first part, the derivation π ends in

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1 & \vdots\,\pi_2 \\ \Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1, \varphi \,;\, \Delta_2 & \psi, \Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2\end{array}}{\varphi \to \psi, \Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2}$$

The inductive hypothesis now gives us four derivable sequents, two for the interpolant χ_1 of the left premise and two for the interpolant χ_2 of the second premise:

$$\begin{array}{llll} \Gamma_1 \Rightarrow \Delta_1, \varphi, \chi_1 & (1) & \chi_1, \Gamma_2 \Rightarrow \Delta_2 & (2) \\ \psi, \Gamma_1 \Rightarrow \Delta_1, \chi_2 & (3) & \chi_2, \Gamma_2 \Rightarrow \Delta_2 & (4) \end{array}$$

Sequents (1) and (3) can serve as premises of →L, if we apply weakening and add χ_2 on the right to (1) and χ_1 on the right to (3):

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ \Gamma_1 \Rightarrow \Delta_1, \varphi, \chi_1, \chi_2 & \psi, \Gamma_1 \Rightarrow \Delta_1, \chi_1, \chi_2\end{array}}{\varphi \to \psi, \Gamma_1 \Rightarrow \Delta_1, \chi_1, \chi_2}\,{\to}\mathrm{L}$$

By applying ∨R we get the sequent

$$\varphi \to \psi, \Gamma_1 \Rightarrow \Delta_1, \chi_1 \vee \chi_2$$

This suggests that $\chi_1 \vee \chi_2$ might be our interpolant in this case. And indeed, we can derive the other required sequent from the remaining sequents (2) and (4):

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ \chi_1, \Gamma_2 \Rightarrow \Delta_2 & \chi_2, \Gamma_2 \Rightarrow \Delta_2\end{array}}{\chi_1 \vee \chi_2, \Gamma_2 \Rightarrow \Delta_2}\,{\vee}\mathrm{L}$$

It remains to verify that $\chi_1 \vee \chi_2$ is in the correct language. The inductive hypothesis gives us that

$$\begin{array}{ll} \mathcal{L}(\chi_1) \subseteq \mathcal{L}(\Gamma_1, \Delta_1, \varphi) \cap \mathcal{L}(\Gamma_2, \Delta_2) & \text{and} \\ \mathcal{L}(\chi_2) \subseteq \mathcal{L}(\psi, \Gamma_1, \Delta_1) \cap \mathcal{L}(\Gamma_2, \Delta_2) & \end{array}$$

Since

$$\begin{aligned}\mathcal{L}(\varphi) &\subseteq \mathcal{L}(\varphi \to \psi) \qquad \text{and}\\ \mathcal{L}(\psi) &\subseteq \mathcal{L}(\varphi \to \psi)\end{aligned}$$

we have both

$$\begin{aligned}\mathcal{L}(\chi_1) &\subseteq \mathcal{L}(\Gamma_1, \Delta_1, \varphi \to \psi) \cap \mathcal{L}(\Gamma_2, \Delta_2) \qquad \text{and}\\ \mathcal{L}(\chi_2) &\subseteq \mathcal{L}(\Gamma_1, \Delta_1, \varphi \to \psi) \cap \mathcal{L}(\Gamma_2, \Delta_2)\end{aligned}$$

and consequently, $\mathcal{L}(\chi_1 \vee \chi_2) =$

$$\mathcal{L}(\chi_1) \cup \mathcal{L}(\chi_2) \subseteq \mathcal{L}(\Gamma_1, \Delta_1, \varphi \to \psi) \cap \mathcal{L}(\Gamma_2, \Delta_2),$$

which is as required.

In the case where $\varphi \to \psi$ belongs to the second part, the situation is different, but handled in a similar way. The inductive hypothesis again gives us four derivable sequents:

$$\begin{array}{llll}\Gamma_1 \Rightarrow \Delta_1, \chi_1 & (1) & \chi_1, \Gamma_2 \Rightarrow \Delta_2, \varphi & (2)\\ \Gamma_1 \Rightarrow \Delta_1, \chi_2 & (3) & \chi_2, \psi, \Gamma_2 \Rightarrow \Delta_2 & (4)\end{array}$$

Now sequents (2) and (4)—plus some weakened formulas—serve as premises of →L:

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ \chi_1, \chi_2, \Gamma_2 \Rightarrow \Delta_2, \varphi & \psi, \chi_1, \chi_2, \Gamma_2 \Rightarrow \Delta_2\end{array}}{\varphi \to \psi, \chi_1, \chi_2, \Gamma_2 \Rightarrow \Delta_2} \to\text{L}$$

We again want to turn this into a sequent that has a single formula built from χ_1 and χ_2, but this time we need it in the antecedent (since we're now dealing with the second part). Applying ∧L does the trick; we should take $\chi \equiv (\chi_1 \wedge \chi_2)$ as the interpolant. Fortuitously, we can use the remaining sequents (1) and (3) to derive the corresponding sequent for the other part:

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ \Gamma_1 \Rightarrow \Delta_1, \chi_1 & \Gamma_1 \Rightarrow \Delta_1, \chi_2\end{array}}{\Gamma_1 \Rightarrow \Delta_1, \chi_1 \wedge \chi_2} \wedge\text{R}$$

As before, we have $\mathcal{L}(\chi_1 \wedge \chi_2) \subseteq \mathcal{L}(\Gamma_1, \Delta_1) \cap \mathcal{L}(\Gamma_2, \Delta_2, \varphi \to \psi)$.

4. π ends in $\forall$R with principal formula $\forall x\,\varphi(x)$:

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\end{array}\\ \Gamma_1\,;\,\Gamma_2 \Rightarrow \Delta_1, \varphi(c)\,;\,\Delta_2}{\Gamma_1\,;\,\Gamma_2 \Rightarrow \Delta_1, \forall x\,\varphi(x)\,;\,\Delta_2}\,\forall\mathrm{R}$$

or

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\end{array}\\ \Gamma_1\,;\,\Gamma_2 \Rightarrow \Delta_1\,;\,\Delta_2, \varphi(c)}{\Gamma_1\,;\,\Gamma_2 \Rightarrow \Delta_1\,;\,\Delta_2, \forall x\,\varphi(x)}\,\forall\mathrm{R}$$

In the first scenario, the induction hypothesis gives us derivations of $\Gamma_1 \Rightarrow \Delta_1, \varphi(c), \chi_1$ and $\chi_1, \Gamma_2 \Rightarrow \Delta_2$. We can apply $\forall$R to the first to obtain $\Gamma_1 \Rightarrow \Delta_1, \forall x\,\varphi(x), \chi_1$. (The eigenvariable condition is satisfied, since it was already satisfied in the $\forall$R rule at the end of π.) Thus χ_1 would be an interpolant if the condition that $\mathcal{L}(\chi_1) \subseteq \mathcal{L}(\Gamma_1, \Delta_1, \forall x\,\varphi(x)) \cap \mathcal{L}(\Gamma_2, \Delta_2)$ is satisfied. By the inductive hypothesis, we have $\mathcal{L}(\chi_1) \subseteq \mathcal{L}(\Gamma_1, \Delta_1, \varphi(c)) \cap \mathcal{L}(\Gamma_2, \Delta_2)$. So we have to worry about c—can c occur in χ_1? It can't. If c occurs in χ_1, we would have both $c \in \mathcal{L}(\Gamma_1, \Delta_1, \varphi(c))$ (which is true) and also $c \in \mathcal{L}(\Gamma_2, \Delta_2)$. But then c would occur in the conclusion of the $\forall$R inference in π, violating the eigenvariable condition.

The other scenario is similar.

5. π ends in $\exists$R with principal formula $\exists x\,\varphi(x)$. We again have two scenarios. We consider the scenario where $\exists x\,\varphi(x)$ belongs to the first part, and leave the other as an exercise.

In the first scenario, π ends in

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\end{array}\\ \Gamma_1\,;\,\Gamma_2 \Rightarrow \Delta_1, \exists x\,\varphi(x), \varphi(t)\,;\,\Delta_2}{\Gamma_1\,;\,\Gamma_2 \Rightarrow \Delta_1, \exists x\,\varphi(x)\,;\,\Delta_2}\,\exists\mathrm{R}$$

The induction hypothesis gives us derivations of $\Gamma_1 \Rightarrow \Delta_1, \exists x\,\varphi(x), \varphi(t), \chi_1$ and $\chi_1, \Gamma_2 \Rightarrow \Delta_2$. We can apply $\exists$R to the first to obtain $\Gamma_1 \Rightarrow \Delta_1, \exists x\,\varphi(x), \chi_1$. Thus χ_1 would again be an interpolant if the condition that $\mathcal{L}(\chi_1) \subseteq \mathcal{L}(\Gamma_1, \Delta_1, \exists x\,\varphi(x)) \cap \mathcal{L}(\Gamma_2, \Delta_2)$ is satisfied. Here the inductive hypothesis gives us $\mathcal{L}(\chi_1) \subseteq \mathcal{L}(\Gamma_1, \Delta_1, \exists x\,\varphi(x), \varphi(t)) \cap \mathcal{L}(\Gamma_2, \Delta_2)$. Now remember that we assumed that there are no function symbols. That means the term t can only be a constant symbol, i.e., $t \equiv b$. If $b \notin \mathcal{L}(\chi_1)$ or $b \in \mathcal{L}(\Gamma_1, \Delta_1, \exists x\,\varphi(x)) \cap \mathcal{L}(\Gamma_2, \Delta_2)$ there is nothing to worry about: we can take χ_1 as the interpolant. But what if $b \in \mathcal{L}(\chi_1)$ and $b \notin \mathcal{L}(\Gamma_1, \Delta_1, \exists x\,\varphi(x)) \cap \mathcal{L}(\Gamma_2, \Delta_2)$? First of all, by replacing all occurrences

of b in χ_1 by a variable y we can write $\chi_1 \equiv \chi_1'[b/y]$ where χ_1' does not contain b. Furthermore, either $b \notin \mathcal{L}(\Gamma_1, \Delta_1, \exists x\, \varphi(x))$ or $b \notin \mathcal{L}(\Gamma_2, \Delta_2)$. We can take $\chi \equiv \forall y\, \chi_1'(y)$ or $\chi \equiv \exists y\, \chi_1'(y)$, respectively. In the first case, we have:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \Rightarrow \Delta_1, \exists x\, \varphi(x), \varphi(b), \chi_1'(b)\end{array}}{\Gamma_1 \Rightarrow \Delta_1, \exists x\, \varphi(x), \chi_1'(b)}\,\exists\mathrm{R}}{\Gamma_1 \Rightarrow \Delta_1, \exists x\, \varphi(x), \forall y\, \chi_1'(y)}\,\forall\mathrm{R}$$

and

$$\dfrac{\begin{array}{c}\vdots\\ \chi_1'(b), \forall y\, \chi_1'(y), \Gamma_2 \Rightarrow \Delta_2\end{array}}{\forall y\, \chi_1'(y), \Gamma_2 \Rightarrow \Delta_2}\,\forall\mathrm{L}$$

The derivations of the top sequents are from the inductive hypothesis (in the second one, we apply admissibility of weakening with $\forall y\, \chi_1'(y)$ in the antecedent). The eigenvariable condition on $\forall$R in the first derivation is satisfied because $b \notin \mathcal{L}(\Gamma_1, \Delta_1, \exists x\, \varphi(x))$, and χ_1' was defined so it does not contain b.

In the second case, we have (from the inductive hypothesis plus admissibility of weakening by $\exists y\, \chi_1'(y)$):

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \Rightarrow \Delta_1, \exists x\, \varphi(x), \varphi(b), \exists y\, \chi_1'(y), \chi_1'(b)\end{array}}{\Gamma_1 \Rightarrow \Delta_1, \exists x\, \varphi(x), \exists y\, \chi_1'(y), \chi_1'(b)}\,\exists\mathrm{R}}{\Gamma_1 \Rightarrow \Delta_1, \exists x\, \varphi(x), \exists y\, \chi_1'(y)}\,\exists\mathrm{R}$$

and

$$\dfrac{\begin{array}{c}\vdots\\ \chi_1'(b), \Gamma_2 \Rightarrow \Delta_2\end{array}}{\exists y\, \chi_1'(y), \Gamma_2 \Rightarrow \Delta_2}\,\exists\mathrm{L}$$

The eigenvariable condition on $\exists$L in the second derivation is satisfied because $b \notin \mathcal{L}(\Gamma_2, \Delta_2)$, and χ_1' was defined so it does not contain b.

The other cases are similary; we leave them as exercises. □

Problem cut.1. Suppose we had a connective $\uparrow$ with rules

$$\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\varphi \uparrow \psi, \Gamma \Rightarrow \Delta}\,\uparrow\mathrm{L} \qquad \dfrac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi \uparrow \psi}\,\uparrow\mathrm{R}$$

Show that Maehara's Lemma still holds by carrying out the cases in the proof of Lemma cut.2 where π ends in one of these rules.

Having established Maehara's Lemma, we can use it to prove Craig's Interpolation Theorem.

Proof of Theorem cut.1. Suppose $\varphi \Rightarrow \psi$ is derivable. Apply Lemma cut.2 to $\varphi\ ;\ \Rightarrow\ ;\psi$ to obtain the interpolant χ. We have $\vdash \varphi \Rightarrow \chi$ and $\vdash \chi \Rightarrow \psi$. □

Suppose Γ is a theory (set of sentences) in a language $\mathcal{L}$ containing an n-place predicate R. We say that Γ *implicitly defines* R, iff

$$\Gamma, \Gamma' \vDash \forall x_1 \ldots \forall x_n \, (R(x_1, \ldots, x_n) \leftrightarrow R'(x_1, \ldots, x_n)),$$

where Γ' is Γ with all occurrences of R replaced by $R' \notin \mathcal{L}$. Γ *explicitly defines* R iff there is some $\varphi(x_1, \ldots, x_n)$ not containing R such that

$$\Gamma \vDash \forall x_1 \ldots \forall x_n \, (R(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n)).$$

Clearly, if Γ explicitly defines R it implicitly defines R. The converse is not obvious, but nevertheless holds:

Lemma cut.3 (Beth Definability Theorem). *If Γ implicitly defines R then it also explicitly defines it.*

Proof. Suppose Γ implicitly defines R. Let R', Γ' be as above and $\mathcal{L}' = \mathcal{L}(\Gamma') = \mathcal{L} \setminus \{R\} \cup \{R'\}$. By hypothesis, we have

$$\Gamma, \Gamma' \Rightarrow \forall x_1 \ldots \forall x_n \, (R(x_1, \ldots, x_n) \leftrightarrow R'(x_1, \ldots, x_n))$$

Let $c_1, \ldots, c_n$ be constant symbols not occurring in Γ. By the invertibility lemma we have

$$\Gamma, \Gamma', R(c_1, \ldots, c_n) \Rightarrow R'(c_1, \ldots, c_n)$$

Apply Maehara's Lemma to

$$\Gamma, R(c_1, \ldots, c_n)\ ;\ \Gamma' \Rightarrow\ ;\ R'(c_1, \ldots, c_n)$$

to find an $\varphi \equiv \varphi(x_1, \ldots, x_n)$ such that

$$\begin{aligned} \Gamma, R(c_1, \ldots, c_n) &\Rightarrow \varphi(c_1, \ldots, c_n) \\ \varphi(c_1, \ldots, c_n), \Gamma' &\Rightarrow R'(c_1, \ldots, c_n) \end{aligned}$$

have derivations. Since $\mathcal{L}(\varphi) = \mathcal{L}_1 \cap \mathcal{L}_2 = \mathcal{L}(\Gamma) \setminus \{R\}$, φ does not contain R. Replace R' everywhere by R in the derivation of the second sequent to get a derivation of

$$\varphi(c_1, \ldots, c_n), \Gamma \Rightarrow R(c_1, \ldots, c_n)$$

Apply →R and ∀R to get a derivation of

$$\Gamma \Rightarrow \forall x_1 \ldots \forall x_n \, (R(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n)).$$

This shows that $\varphi(x_1, \ldots, x_n)$ explicitly defines ρ in Γ. □

A third result which is equivalent to interpolation (and, like it, can be proved using Maehara's Lemma) is the following theorem: If two consistent theories Γ_1, Γ_2 contain a complete theory Γ in the language $\mathcal{L}(\Gamma) \subseteq \mathcal{L}(\Gamma_1) \cap \mathcal{L}(\Gamma_2)$, then $\Gamma_1 \cup \Gamma_2$ is consistent.

Recall that a complete theory is one such that for every sentence φ in its language either $\Gamma \vdash \varphi$ or $\Gamma \vdash \neg\varphi$. Since Γ_1 and Γ_2 are consistent extensions of Γ, Γ must be consistent. It follows that if φ is a sentence in the language $\mathcal{L}(\Gamma_1) \cap \mathcal{L}(\Gamma_2)$, we can't have $\Gamma_1 \models \varphi$ and $\Gamma_2 \models \neg\varphi$. To see this, suppose there were such an φ. If $\Gamma_1 \models \varphi$ then $\Gamma \models \varphi$. Otherwise, by completeness of Γ, $\Gamma \models \neg\varphi$, and since $\Gamma \subseteq \Gamma_1$, $\Gamma_1 \models \neg\varphi$, making Γ_1 inconsistent. So $\Gamma \models \varphi$ and since $\Gamma \subseteq \Gamma_2$, $\Gamma_2 \models \varphi$. So $\Gamma \not\models \neg\varphi$, or Γ_2 would be inconsistent.

The non-existence of a sentence φ in the joint language $\mathcal{L}(\Gamma_1) \cap \mathcal{L}(\Gamma_2)$ with $\Gamma_1 \models \varphi$ and $\Gamma_2 \models \neg\varphi$ is all that's needed for the proof, which we give here proof-theoretically using Maehara's Lemma.

Theorem cut.4 (Robinson's Joint Consistency Theorem). *Suppose Γ_1, Γ_2 are consistent theories and for no φ in the language $\mathcal{L}(\Gamma_1) \cap \mathcal{L}(\Gamma_2)$ do we have $\Gamma_1 \models \varphi$ and $\Gamma_2 \models \neg\varphi$. Then $\Gamma_1 \cup \Gamma_2$ is consistent.*

Proof. Suppose for the contrary that $\Gamma_1 \cup \Gamma_2$ is inconsistent. Then the sequent $\Gamma_1, \Gamma_2 \Rightarrow$ has a derivation. Apply Maehara's Lemma to $\Gamma_1 \,;\, \Gamma_2 \Rightarrow \;;$ to get a sentence φ in the language $\mathcal{L}(\Gamma_1) \cap \mathcal{L}(\Gamma_2)$ such that $\vdash \Gamma_1 \Rightarrow \varphi$ and $\vdash \varphi, \Gamma_2 \Rightarrow$. From the latter we get $\vdash \Gamma_2 \Rightarrow \neg\varphi$. But we assumed that no such sentence exists. □

Above we have tacitly assumed that the theories Γ, Γ_1, Γ_2 are finite. By compactness, the results also hold for infinite theories.

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Bibliography