Chapter udf

Cut Elimination

cut.1 Introduction

pt:cut:int:sec

The Cut rule is the rule

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\ \text{Cut}$$

(The formula φ is called the *cut formula*.)

Cut was included in Gentzen's original sequent calculus **LK**. Gentzen's *Hauptsatz* ("main theorem") states that every sequent derivable in **LK** is also derivable without using the Cut rule, i.e., that Cut can be "eliminated" from **LK**-proofs. Our systems **G1c** and **G3c** do not contain Cut, but the question can be raised for them as well: can Cut be eliminated from derivations that use Cut plus the other rules of **G1c** (or those of **G3c**)? In our established terminology it can be asked equivalently as: Is Cut an admissible rule of **G1c** (or **G3c**).

The cut elimination theorem is perhaps the most important and celebrated result of proof theory. It first of all establishes that **G1c** and **G3c** don't need a rule which you might, at first blush, think is clearly required. For when you try to formalize actual proofs, you'll notice that you need a way to deal with the use of lemmas. Mathematicians like to prove results, and then appeal to these results in proofs of other results. So you might first formalize a proof of result L (the lemma) as the derivation of a sequent $\Rightarrow L$. Then you formalize the proof of the second result R, the proof of which uses L, as the derivation of a sequent $L \Rightarrow R$. How do you know that there is a proof of R, i.e., a derivation of just $\Rightarrow R$? You need a way to combine the two derivations, and the Cut rule would allow you to do that.

We mentioned that **G1c** and **G3c** are complete, and so they prove any sequent you might expect to be able to prove—all valid sequents. This can be proved directly. But at the time Gentzen was writing, only axiomatic derivation systems were known to be complete. To show **LK** complete, Gentzen provided a translation of derivations in the axiomatic system into **LK**-derivations. This

translation used Cut essentially. Cut elimination is not just an important result for classical logic: many other logics have sequent calculi. For some logics, a direct proof of completeness for their sequent calculi is hard or impossible, so translations from other systems (using Cut) are the only way to show completeness. Then proving cut elimination proof-theoretically is necessary to establish that Cut is superfluous and that the system without Cut is complete.

Cut elimination is also important because it can be applied to obtain other, independently interesting results. Two cases we'll consider are the Craig interpolation lemma and Herbrand' theorem.

The ideas used to prove cut elimination typically involve showing how a derivation that uses Cut can be transformed into one without. These transformations are usually "local," i.e., we take a part of a derivation (usually a sub-derivation ending in a Cut inference) and replace it with another subderivation of the same sequent but in which the Cut inference has been removed or replaced by others. Such arguments provide step-by-step algorithms for transforming derivations with Cut into ones without. This provides more information when cut elimination procedures are used in applications. For instance, there are model theoretic proofs of the interpolation lemma the form "if $\varphi \to \psi$" is valid then an interpolant exists (never mind what and interpolant is for the moment). A proof-theoretic argument using cut elimination provides an algorithm that takes a derivation of $\varphi \to \psi$ and returns an interpolant. In contrast to the model theoretic argument, the proof theoretic argument is *constructive*.

There are two popular ways to prove the cut elimination theorem. One (going back to Gentzen) shows that you can remove top-most cuts in a proof, and then just removes top-most cuts until none are left. The other (originally due to Schütte and Tait) focuses on the most complex cuts in a derivation, and successively removes cuts that are maximal in the sense that no other cut has a cut formula of higher depth. They each have their advantages and drawbacks. For some systems, one approach works while another is hard to implement or even impossible. So, we describe both approaches. We'll prove cut elimination for **G3c**. In fact, we'll show not that Cut can be eliminated, but *context-sharing cut* $\mathrm{Cut}_{\mathrm{CS}}$:

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}$$

If a sequent calculus allows weakening and contraction (either as actual rules like in **G1c** or as admissible rules like in **G3c**), Cut can simulate $\mathrm{Cut}_{\mathrm{CS}}$, and vice versa. We choose $\mathrm{Cut}_{\mathrm{CS}}$ over Cutbecause the first cut elimination strategy is easier to describe for $\mathrm{Cut}_{\mathrm{CS}}$, and the second strategy only works for $\mathrm{Cut}_{\mathrm{CS}}$.

cut.2 Removing Topmost Cuts

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Definition cut.1. Consider a derivation π ending in a $\mathrm{Cut}_{\mathrm{CS}}$ rule,

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1 & \vdots\,\pi_2\\ \Gamma \Rightarrow \Delta, \varphi & \varphi, \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

and containing no other $\mathrm{Cut}_{\mathrm{CS}}$ inferences. The *cut height* of π is $\mathrm{ch}(\pi) = \mathrm{ht}(\pi_1) + \mathrm{ht}(\pi_2)$. The *cut rank* of π is $\mathrm{cr}(\pi) = \mathrm{dp}(\varphi)$.

pt:cut:top: lem:cut-adm-G3c

Lemma cut.2. *The* $\mathrm{Cut}_{\mathrm{CS}}$ *rule is admissible in* **G3c**. *In other words, if* π *is a derivation ending in a single* $\mathrm{Cut}_{\mathrm{CS}}$ *and otherwise using only rules of* **G3c**, *then there is a derivation* π' *of the same end-sequent in* **G3c**.

Proof. The proof is by double induction on cut height and rank. The derivation π ends in:

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1 & \vdots\,\pi_2\\ \Gamma \Rightarrow \Delta, \varphi & \varphi, \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

Induction basis: $\mathrm{ch}(\pi) = 0$ and $\mathrm{cr}(\pi) = 0$. Since $\mathrm{ch}(\pi) = \mathrm{ht}(\pi_1) + \mathrm{ht}(\pi_2) = 0$, both $\Gamma \Rightarrow \Delta, \varphi$ and $\varphi, \Gamma \Rightarrow \Delta$ are axioms. There are two cases: (a) either φ is not principal in one of them, or (b) φ is principal in both of them. In both cases we'll show below that $\Gamma \Rightarrow \Delta$ must be an axiom (and we don't need the inductive hypothesis to do that).

We'll now distinguish a number of possible cases according to whether π_1 and π_2 are axioms or end in an inference, and whether φ is principal in that inference or not. Cases A and B cover the induction basis. In the inductive step we assume $\mathrm{ch}(\pi) > 0$ or $\mathrm{cr}(\pi) > 0$ (or both). We can use the inductive hypothesis, namely that the last $\mathrm{Cut}_{\mathrm{CS}}$ can be removed from any derivation which ends in a single $\mathrm{Cut}_{\mathrm{CS}}$ which either has cut rank $= \mathrm{cr}(\pi)$ and a cut height $< \mathrm{ch}(\pi)$, or has cut rank $< \mathrm{cr}(\pi)$. The case where $\mathrm{ch}(\pi) = 0$ (both premises are axioms) are covered by cases A and B. The cases where $\mathrm{ch}(\pi) > 0$ are covered by cases C–D.

Case A: Some ψ occurs in both Γ and Δ. Then $\Gamma \Rightarrow \Delta$ is an axiom (if ψ is atomic), or has a derivation π' in **G3c** without $\mathrm{Cut}_{\mathrm{CS}}$ by **??**. This covers case (a) of the induction basis, since if φ is not principal in $\Gamma \Rightarrow \Delta, \varphi$ or not principal in $\varphi, \Gamma \Rightarrow \Delta$ and these are axioms, then some other atomic ψ must occur in both Γ and Δ. It also covers the inductive step if one of the premises is an axiom in which φ is not principal (even if the other premise is the conclusion of a rule).

Case B: Both premises are axioms, and φ is principal and hence atomic. Consider the left premise, $\Gamma \Rightarrow \Delta, \varphi$. Since φ is principal, it must occur in Γ (otherwise $\Gamma \Rightarrow \Delta, \varphi$ wouldn't be an axiom). Similarly, φ must occur in Δ, or else $\varphi, \Gamma \Rightarrow \Delta$ wouldn't be an axiom. Thus φ is atomic and occurs in both Γ and Δ, i.e., $\Gamma \Rightarrow \Delta$ is an axiom. This covers case (b) of the induction basis.

Alternative cases A and B

Case A: One of the premises of the cut is an axiom of the form $\chi, \Gamma' \Rightarrow \Delta', \chi$ with χ atomic. If χ is not the cut formula φ, then χ must occur in both Γ and Δ. In that case, $\Gamma \Rightarrow \Delta$ is an axiom and we can take it as π'. If $\chi \equiv \varphi$ is the cut formula then either (if the left premise is an axiom) $\varphi \in \Gamma$ and $\Gamma = \Gamma', \varphi$, or (if the right premise is an axiom) $\varphi \in \Delta$ and $\Delta = \Delta', \varphi$. In the first case, π_2 is a derivation of $\varphi, \varphi, \Gamma' \Rightarrow \Delta$ and by admissibility of CL (**??**) there is a derivation of $\Gamma \Rightarrow \Delta$. In the second case, π_2 is a derivation of $\Gamma \Rightarrow \Delta', \varphi, \varphi$ and by admissibility of CR we get a derivation of $\Gamma \Rightarrow \Delta$.

Case B: One of the premises is an axiom of the form $\bot, \Gamma' \Rightarrow \Delta'$. If the left premise is such an axiom, then $\bot \in \Gamma$ and $\Gamma \Rightarrow \Delta$ is also an axiom. If the right premise is such an axiom, then either $\bot \in \Gamma$ (in which case $\Gamma \Rightarrow \Delta$ again is an axiom), or $\varphi \equiv \bot$. In the latter case, π_1 is a derivation of $\Gamma \Rightarrow \Delta, \bot$. We can turn it into a derivation of $\Gamma \Rightarrow \Delta$ by deleting one copy of $\bot$ from the consequent of every sequent in π_1. (Exercise: verify this by induction on $\mathrm{ht}(\pi_1)$.)

Now suppose that neither case A nor case B applies. In the remaining cases, $\mathrm{ch}(\pi) > 0$, i.e., at least one of the premises is the conclusion of an inference.

Cases C and D: Permuting cuts. The possibilities considered in cases C and D all have a common pattern. We begin with a derivation ending in $\mathrm{Cut_{CS}}$ where one premise is obtained by a rule R in which the cut formula φ is not principal (some other formula θ is). In this derivation, $\mathrm{Cut_{CS}}$ follows the rule R, and schematically looks as follows:

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \Gamma \Rightarrow \Delta', \theta, \varphi\end{array} \qquad \left.\dfrac{\begin{array}{c}\vdots\,\pi_3\\ \varphi, \Gamma, \Pi \Rightarrow \Delta', \Lambda\end{array}}{\varphi, \Gamma \Rightarrow \Delta', \theta}\, R\right\}\pi_2}{\Gamma \Rightarrow \Delta', \theta}\,\mathrm{Cut_{CS}}$$

This only covers the case where R is a right rule with one premise, A is not the principal formula of R, and the formula θ which *is* principal occurs in the succedent of the right premise of $\mathrm{Cut_{CS}}$. If θ occurs in the antecedent (i.e., R is a left rule) or in the left premise of $\mathrm{Cut_{CS}}$, it would look different, of course. The formulas in Π and Λ represent the active formulas of rule R.

We turn this order around and consider a derivation where R follows $\mathrm{Cut}_{\mathrm{CS}}$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi_1'\end{array}}{\Gamma, \Pi \Rightarrow \Delta', \theta, \Lambda, \varphi} \qquad \dfrac{\begin{array}{c}\vdots\\ \pi_3'\end{array}}{\varphi, \Gamma, \Pi \Rightarrow \Delta', \theta, \Lambda}}{\dfrac{\Gamma, \Pi \Rightarrow \Delta', \theta, \Lambda}{\Gamma, \Rightarrow \Delta', \theta, \theta}\, R}\, \mathrm{Cut}_{\mathrm{CS}}$$

We use height-preserving weakening to obtain π_1' and π_3' from π_1 and π_3, respectively.

Because we switch the order of $\mathrm{Cut}_{\mathrm{CS}}$ and R, this is referred to as "permuting a cut upwards across R." Doing so decreases the height of the derivation ending in the right premise of $\mathrm{Cut}_{\mathrm{CS}}$, and so decreases the cut height. That is, the sub-derivations ending in the new $\mathrm{Cut}_{\mathrm{CS}}$ can be assumed to have height no larger than π_1 and π_3, so the cut height is reduced (we've removed one inference on the right). Now the induction hypothesis applies, which tells us that the $\mathrm{Cut}_{\mathrm{CS}}$ inference can be removed. Finally, we remove the extra copy of θ using the admissibility of CR.

Case C: At least one of π_1 and π_2 ends in a one-premise rule R in which φ is not principal. There are a total of 18 cases, depending on whether φ is not principal in the left or the right premise of $\mathrm{Cut}_{\mathrm{CS}}$, and depending on which of the nine one-premise rules ($\neg$L, $\neg$R, $\vee$R, $\wedge$L, $\to$R, and the four quantifier rules) the rule R is. We only consider two examples: where φ is not principal in the last inference of π_2, which ends in $\to$R or in $\exists$L. The other cases are analogous and left as exercises.

Suppose π ends in:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi_1\end{array}}{\Gamma \Rightarrow \Delta', \psi \to \chi, \varphi} \qquad \left.\dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi_3\end{array}}{\varphi, \psi, \Gamma \Rightarrow \Delta', \chi}}{\varphi, \Gamma \Rightarrow \Delta', \psi \to \chi}\, \to\mathrm{R}\right\}\pi_2}{\Gamma \Rightarrow \Delta', \psi \to \chi}\, \mathrm{Cut}_{\mathrm{CS}}$$

where $\Delta = \Delta', \psi \to \chi$. We apply height-preserving weakening (**??**) to π_1 to obtain a derivation π_1' of $\psi, \Gamma \Rightarrow \Delta', \psi \to \chi, \chi, \varphi$ (we weaken with ψ on the left and χ on the right). Similarly, we apply **??** to π_3 to get a derivation π_3' of $\varphi, \psi, \Gamma \Rightarrow \Delta', \psi \to \chi, \chi$ (by weakening with $\psi \to \chi$ on the right). The induction hypothesis applies to

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi_1'\end{array}}{\psi, \Gamma \Rightarrow \Delta', \psi \to \chi, \chi, \varphi} \qquad \dfrac{\begin{array}{c}\vdots\\ \pi_3'\end{array}}{\varphi, \psi, \Gamma \Rightarrow \Delta', \psi \to \chi, \chi}}{\psi, \Gamma \Rightarrow \Delta', \psi \to \chi, \chi}\, \mathrm{Cut}_{\mathrm{CS}}$$

with cut formula φ: This derivation has the same cut rank and a lower cut height than π. This gives us a derivation π_4 in **G3c** without $\mathrm{Cut}_{\mathrm{CS}}$ of $\psi, \Gamma \Rightarrow \Delta', \psi \to \chi, \chi$. Apply $\to$R to get a derivation of $\Gamma \Rightarrow \Delta', \psi \to \chi, \psi \to \chi$:

$$\dfrac{\begin{array}{c}\vdots\\ \pi_4\\ \vdots\end{array}}{\psi, \Gamma \Rightarrow \Delta', \psi \to \chi, \chi}{\Gamma \Rightarrow \Delta', \psi \to \chi, \psi \to \chi}\,{\to}\mathrm{R}$$

To get a derivation π' of $\Gamma \Rightarrow \Delta$, i.e., $\Gamma \Rightarrow \Delta', \psi \to \chi$, apply **??** (the admissibility of contraction).

Now suppose π ends in:

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\end{array}}{\exists x\, \psi(x), \Gamma' \Rightarrow \Delta, \varphi} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi_3\\ \vdots\end{array}}{\varphi, \psi(c), \Gamma' \Rightarrow \Delta}}{\varphi, \exists x\, \psi(x), \Gamma' \Rightarrow \Delta}\,\exists\mathrm{L}\Bigg\}\pi_2}{\Gamma' \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

We again apply the inductive hypothesis to

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi_1[\psi(c) \Rightarrow\,]\\ \vdots\end{array}}{\exists x\, \psi(x), \psi(c), \Gamma' \Rightarrow \Delta, \varphi} \qquad \dfrac{\begin{array}{c}\vdots\\ \pi_3[\exists x\, \psi(x) \Rightarrow\,]\\ \vdots\end{array}}{\varphi, \exists x\, \psi(x), \psi(c), \Gamma' \Rightarrow \Delta}}{\exists x\, \psi(x), \psi(c), \Gamma' \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}}{\exists x\, \psi(x), \exists x\, \psi(x), \Gamma' \Rightarrow \Delta}\,\exists\mathrm{L}$$

Note that the eigenvariable condition is satisfied, since the eigenvariable condition on $\exists$L in π_2 guarantees that c does not occur in $\exists x\, \psi(x)$, Γ', or Δ. Finally, apply **??**.

Case D: At least one of π_1 and π_2 ends in a two-premise rule R in which φ is not principal. There are 6 cases. We consider the case where φ is not principal in the last inference of π_2, which ends in $\wedge$R; the other cases are analogous.

Assume π ends in:

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\end{array}}{\Gamma \Rightarrow \Delta', \psi \wedge \chi, \varphi} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi_3\\ \vdots\end{array}}{\varphi, \Gamma \Rightarrow \Delta', \psi} \qquad \dfrac{\begin{array}{c}\vdots\\ \pi_4\\ \vdots\end{array}}{\varphi, \Gamma \Rightarrow \Delta', \chi}\,\wedge\mathrm{R}\Bigg\}\pi_2}{\varphi, \Gamma \Rightarrow \Delta', \psi \wedge \chi}\,\mathrm{Cut}_{\mathrm{CS}}}{\Gamma \Rightarrow \Delta}$$

The inductive hypothesis applies to the derivations

$$\dfrac{\begin{array}{c}\vdots\,\pi_1'\\ \Gamma \Rightarrow \Delta', \psi \wedge \chi, \psi, \varphi\end{array} \qquad \begin{array}{c}\vdots\,\pi_3'\\ \varphi, \Gamma \Rightarrow \Delta', \psi \wedge \chi, \psi\end{array}}{\Gamma \Rightarrow \Delta', \psi \wedge \chi, \psi}\,\mathrm{Cut_{CS}}$$

and

$$\dfrac{\begin{array}{c}\vdots\,\pi_1''\\ \Gamma \Rightarrow \Delta, \psi \wedge \chi, \chi, \varphi\end{array} \qquad \begin{array}{c}\vdots\,\pi_4'\\ \varphi, \Gamma \Rightarrow \Delta', \psi \wedge \chi, \chi\end{array}}{\Gamma \Rightarrow \Delta', \psi \wedge \chi, \chi}\,\mathrm{Cut_{CS}}$$

Here, the **G3c** derivations π_1' and π_1'' are obtained by height-preserving weakening (**??**) from π_1 (once weakening on the right with ψ, and once with χ). The derivations π_3' and π_4' are obtained from π_3 and π_4 also by height-preserving weakening from π_3 and π_4, respectively (by weakening with $\psi \wedge \chi$ on the right).

The inductive hypothesis gives us derivations π_5 and π_6 in **G3c** of $\Gamma \Rightarrow \Delta', \psi \wedge \chi, \psi$ and $\Gamma \Rightarrow \Delta', \psi \wedge \chi, \chi$. We combine these into a derivation of $\Gamma \Rightarrow \Delta', \psi \wedge \chi, \psi \wedge \chi$ using the rule $\wedge$R:

$$\dfrac{\begin{array}{c}\vdots\,\pi_5\\ \Gamma \Rightarrow \Delta', \psi \wedge \chi, \psi\end{array} \qquad \begin{array}{c}\vdots\,\pi_6\\ \Gamma \Rightarrow \Delta', \psi \wedge \chi, \chi\end{array}}{\Gamma \Rightarrow \Delta, \psi \wedge \chi, \psi \wedge \chi}\,\wedge\mathrm{R}$$

To get a derivation of $\Gamma \Rightarrow \Delta$, i.e., $\Gamma \Rightarrow \Delta', \psi \wedge \chi$, apply **??** (the admissibility of contraction).

Case E: Reducing cuts. The final case concerns the situation where both π_1 and π_2 end in inferences in which φ is principal. There are six cases, one for each possible main operator of φ.

In each case, we turn a $\mathrm{Cut_{CS}}$ with cut formula φ into one or more cuts on sub-formulas of φ, namely on the active formulas of the inferences of which φ is the principal formula. These cases are therefore often referred to as "reduction" or "conversion" of the cut inference.

1. If $\varphi \equiv \neg\psi$ then π ends in

$$\pi_1\left\{\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1'\\ \psi, \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta, \neg\psi}\,\neg\mathrm{R} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_2'\\ \Gamma \Rightarrow \Delta, \psi\end{array}}{\neg\psi, \Gamma \Rightarrow \Delta}\,\neg\mathrm{L}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut_{CS}}\right\}\pi_2$$

By inductive hypothesis, $\mathrm{Cut}_{\mathrm{CS}}$ can be eliminated from

$$\dfrac{\begin{array}{cc}\vdots\,\pi_2' & \vdots\,\pi_1' \\ \Gamma \Rightarrow \Delta, \psi & \psi, \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

2. If $\varphi \equiv (\psi \vee \chi)$ then π ends in

$$\pi_1\left\{\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1' \\ \Gamma \Rightarrow \Delta, \psi, \chi\end{array}}{\Gamma \Rightarrow \Delta, \psi \vee \chi}\vee\mathrm{R} \qquad \dfrac{\begin{array}{ccc}\vdots\,\pi_2' & & \vdots\,\pi_2'' \\ \psi, \Gamma \Rightarrow \Delta & & \chi, \Gamma \Rightarrow \Delta\end{array}}{\psi \vee \chi, \Gamma \Rightarrow \Delta}\vee\mathrm{L}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}\right\}\pi_2$$

Consider the derivation ending in $\mathrm{Cut}_{\mathrm{CS}}$,

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1' & \vdots\,\pi_2''' \\ \Gamma \Rightarrow \Delta, \psi, \chi & \chi, \Gamma \Rightarrow \Delta, \psi\end{array}}{\Gamma \Rightarrow \Delta, \psi}\,\mathrm{Cut}_{\mathrm{CS}}$$

where π_2''' is obtained from π_2'' by height-preserving weakening. It has cut rank $<$ $\mathrm{cr}(\pi)$ as $\mathrm{dp}(\chi) < \mathrm{dp}(\psi \vee \chi)$, so the inductive hypothesis yields a derivation in **G3c** of π_1'' of $\Gamma \Rightarrow \Delta, \psi$. The derivation ending in $\mathrm{Cut}_{\mathrm{CS}}$,

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1'' & \vdots\,\pi_2' \\ \Gamma \Rightarrow \Delta, \psi & \psi, \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

has cut rank $= \mathrm{dp}(\psi) < \mathrm{dp}(\psi \vee \chi)$, so the inductive hypothesis applies again, and yields a derivation π' of $\Gamma \Rightarrow \Delta$.

3. $\varphi \equiv (\psi \wedge \chi)$. Exercise.

4. If $\varphi \equiv (\psi \to \chi)$ then π ends in

$$\pi_1\left\{\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1' \\ \psi, \Gamma \Rightarrow \Delta, \chi\end{array}}{\Gamma \Rightarrow \Delta, \psi \to \chi}\to\mathrm{R} \qquad \dfrac{\begin{array}{ccc}\vdots\,\pi_2' & & \vdots\,\pi_2'' \\ \Gamma \Rightarrow \Delta, \psi & & \chi, \Gamma \Rightarrow \Delta\end{array}}{\psi \to \chi, \Gamma \Rightarrow \Delta}\to\mathrm{L}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}\right\}\pi_2$$

The derivation ending in $\mathrm{Cut}_{\mathrm{CS}}$,

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1' & \vdots\,\pi_2'' \\ \psi, \Gamma \Rightarrow \Delta, \chi & \chi, \Gamma \Rightarrow \Delta\end{array}}{\psi, \Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

has lower cut rank than π. The inductive hypthesis yields a derivation π_1'' of $\psi, \Gamma \Rightarrow \Delta$. Now consider the derivation ending in $\mathrm{Cut}_{\mathrm{CS}}$,

$$\dfrac{\begin{array}{cc}\vdots\,\pi_2' & \vdots\,\pi_1'' \\ \Gamma \Rightarrow \Delta, \psi & \psi, \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

Its cut rank is also less than the cut rank of π, so the induction hypothesis yields a **G3c**-derivation π' of $\Gamma \Rightarrow \Delta$.

5. If $\varphi \equiv \forall x\, \psi(x)$ then π ends in

$$\dfrac{\left\{\dfrac{\dfrac{\vdots\,\pi_1'}{\Gamma \Rightarrow \Delta, \psi(c)}}{\Gamma \Rightarrow \Delta, \forall x\, \psi(x)}\,\forall\mathrm{R} \quad \dfrac{\dfrac{\vdots\,\pi_2'}{\psi(t), \forall x\, \psi(x), \Gamma \Rightarrow \Delta}}{\forall x\, \psi(x), \Gamma \Rightarrow \Delta}\,\forall\mathrm{L}\right\}\pi_2}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

By inductive hypothesis, $\mathrm{Cut}_{\mathrm{CS}}$ can be eliminated from

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1'' & \vdots\,\pi_2' \\ \psi(t), \Gamma \Rightarrow \Delta, \forall x\, \psi(x) & \psi(t), \forall x\, \psi(x), \Gamma \Rightarrow \Delta\end{array}}{\psi(t), \Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

where π_1'' is obtained from π_1 (not π_1'!) by height-preserving weakening. (The derivation has the same cut rank but lower cut height.) This gives us a derivation π_2'' of $\psi(t), \Gamma \Rightarrow \Delta$.

The derivation π_1' has end-sequent $\Gamma \Rightarrow \Delta, \psi(c)$, where c is the eigenvariable of a $\forall$R inference. So c does not occur in $\psi(x)$, Γ, or Δ. We assume the derivation π is regular, so c is the eigenvariable of only the following $\forall$R inference and only occurs above it, i.e., only occurs in π_1' and is not an eigenvariable of an inference in π_1'. Since c does not occur in π_2, it cannot occur in t. By **??**, the derivation $\pi_1'[t/c]$ is a derivation of $\Gamma \Rightarrow \Delta, \psi(t)$. We now apply the inductive hypothesis to the derivation

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1'[t/c] & \vdots\,\pi_2'' \\ \Gamma \Rightarrow \Delta, \psi(t) & \psi(t), \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

(The induction hypothesis applies since the cut formula is $\psi(t)$, so the derivation has lower cut rank than π.)

6. We leave the case $\varphi \equiv \exists x\, \psi(x)$ as an exercise. □

Problem cut.1. Verify the sub-case of case D in the proof of Lemma cut.2 where φ is not principal in the last inference of π_1, which ends in $\to$L. (Note: part of this exercise is to clearly state what you have to prove, i.e., what form π has in this case.)

Problem cut.2. Verify the sub-case of case D in the proof of Lemma cut.2 where φ is not principal in the last inference of π_1, which ends in $\forall$L.

Problem cut.3. Verify the sub-case of case E in the proof of Lemma cut.2 where φ is principal in the last inferences of both π_1 and π_2 and $\varphi \equiv (\psi \wedge \chi)$.

Problem cut.4. Verify the sub-case of case E in the proof of Lemma cut.2 where φ is principal in the last inferences of both π_1 and π_2 and $\varphi \equiv \exists x\, \psi(x)$.

Note that in case E the cut height of the derivation ending in Cut_{CS} to which we apply the inductive hypothesis can be larger than the cut height of the original proof. E.g., in the case where $\varphi \equiv (\psi \to \chi)$ we turned π into a derivation involving two Cut_{CS} inferences,

$$\dfrac{\dfrac{\vdots\,\pi_2'}{\Gamma \Rightarrow \Delta, \psi} \qquad \dfrac{\dfrac{\vdots\,\pi_1'}{\psi, \Gamma \Rightarrow \Delta, \chi} \qquad \dfrac{\vdots\,\pi_2''}{\chi, \Gamma \Rightarrow \Delta}\,\text{Cut}_{\text{CS}}}{\psi, \Gamma \Rightarrow \Delta}\,\text{Cut}_{\text{CS}}}{\psi, \Gamma \Rightarrow \Delta}\;\Big\}\,\pi_1''$$

The top Cut_{CS} between π_1' and π_2'' has both lower cut rank and lower cut height than π. But the derivation π_1''—the result of applying the inductive hypothesis—no longer has lower cut height than π. Since the cut formula of the lower Cut_{CS} is ψ, its cut *rank*, however, is lower than the cut rank of π.

Lemma cut.2 establishes that we can eliminate a single topmost Cut_{CS} inference from a derivation. We can show that we can eliminate any number of Cut_{CS} inferences from a derivation by successively applying it to the topmost sub-derivations ending in Cut_{CS}.

Theorem cut.3. *Any derivation π in* **G3c** $+ \text{Cut}_{\text{CS}}$ *can be transformed into a derivation in* **G3c**. *pt:cut:top: thm:cut-elim-G3c-topmost*

Proof. By induction on the number n of Cut_{CS} inferences in π. If $n = 0$, there is nothing to do: π is already a derivation in **G3c**. Otherwise, consider the sub-derivation π' ending in a top-most Cut_{CS} inference. It contains a single Cut_{CS} rule as the last inference. Apply Lemma cut.2 to transform it into a Cut_{CS}-free proof π'', and replace the sub-derivation π' by π''. The result is a derivation containing $n - 1$ Cut_{CS} inferences, and the induction hypothesis applies. □

cut.3 Removing Largest Cuts

pt:cut:inv: sec

The proof of **??** shown that if **G3c** derives the conclusion of a logical rule (other than ∃R and ∀L), it also derives each of the premises of the rule without increasing the height of the proof. We can do better and show that this holds also for $\mathbf{G3c} + \mathrm{Cut_{CS}}$.

As before, we take the *cut rank* of a $\mathrm{Cut_{CS}}$ inference to be the depth of its cut formula.

pt:cut:inv: lem:inv-G3c-cut

Lemma cut.4. *If* $\mathbf{G3c} + \mathrm{Cut_{CS}}$ *derives the conclusion of a logical rule* R *other than* ∃R *or* ∀L *using a derivation* π *then it also derives each premise with derivation* π' *(or derivations* π'*,* π'' *depending on whether the rule has one or two premises). Moreover, (a)* $\mathrm{ht}(\pi') \leq \mathrm{ht}(\pi)$ *(and* $\mathrm{ht}(\pi'') \leq \mathrm{ht}(\pi)$*) and (b) the number of* $\mathrm{Cut_{CS}}$ *inferences as well as their ranks is the same in* π' *(and* π''*) as in* π*.*

Proof. By inspection of the proofs of **????**. That proof was by induction on $\mathrm{ht}(\pi)$. In the base case, we replaced an axiom by another axiom. So conditions (a) and (b) are trivially satisfied. If the last rule of π was R, its immediate sub-derivations are the derivations π_i we want, and (a) and (b) are both satisfied. In all other cases, we applied the inductive hypothesis to the immediate sub-derivations of π, i.e., those leading to the premise(s) of the last inference R, and then applied that same rule R to the results. The conditions (a) and (b) hold for the immediate sub-derivations of the new proof, and so (a) and (b) hold for the entire proof. It remains to verify the case where the last inference is $\mathrm{Cut_{CS}}$. We do this again for one example only: suppose π is a derivation of the conclusion $\psi \wedge \chi, \Gamma \Rightarrow \Delta$ of ∧L, and ends in $\mathrm{Cut_{CS}}$:

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1 & \vdots\,\pi_2\\ \psi\wedge\chi,\Gamma \Rightarrow \Delta,\varphi & \varphi,\psi\wedge\chi,\Gamma\Rightarrow\Delta\end{array}}{\psi\wedge\chi,\Gamma\Rightarrow\Delta}\,\mathrm{Cut_{CS}}$$

The induction hypothesis applies to π_1 and π_2 (which are also derivations of instances of the conclusion of ∧L) and yields derivations π_1' and π_2' of $\psi, \chi, \Gamma \Rightarrow \Delta, \varphi$ and $\varphi, \psi, \chi, \Gamma \Rightarrow \Delta$, respectively. We combine these using $\mathrm{Cut_{CS}}$ to obtain π':

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1' & \vdots\,\pi_2'\\ \psi,\chi,\Gamma \Rightarrow \Delta,\varphi & \varphi,\psi,\chi,\Gamma\Rightarrow\Delta\end{array}}{\psi,\chi,\Gamma\Rightarrow\Delta}\,\mathrm{Cut_{CS}}$$

Clearly, (a) and (b) are satisfied. □

Note that this would not work if we used Cut instead of $\mathrm{Cut_{CS}}$, as then the conclusion of the last derivation would have two copies of ψ, χ, Γ in the antecedent and two copies of Δ in the succedent. We would have to appeal

to the admissibility of contraction, but the proof of **??** uses the invertibility lemma. We'd thus be arguing in a circle.

We can use Lemma cut.4 to give an alternative proof of cut elimination. In this proof we do not successively remove topmost occurrences of $\mathrm{Cut}_{\mathrm{CS}}$ inferences, but occurrences of $\mathrm{Cut}_{\mathrm{CS}}$ inferences of maximal rank. That is, we start with a derivation π in $\mathbf{G3c} + \mathrm{Cut}_{\mathrm{CS}}$ and remove a cut anywhere in it (possibly replacing it with cuts of smaller rank)—and then keep doing that until no cuts remain. The only condition is that above the cut we deal with, no cuts of the same or higher rank occur.

pt:cut:inv:
lem:max-cut-red-G3c

Lemma cut.5. *If π is a derivation in $\mathbf{G3c} + \mathrm{Cut}_{\mathrm{CS}}$ ending in a $\mathrm{Cut}_{\mathrm{CS}}$ inference of rank n and otherwise using only rules of $\mathbf{G3c}$ and $\mathrm{Cut}_{\mathrm{CS}}$ inferences of rank $< n$, then there is a proof π' of the same end-sequent in $\mathbf{G3c} + \mathrm{Cut}_{\mathrm{CS}}$ where every $\mathrm{Cut}_{\mathrm{CS}}$ inference has rank $< n$.*

Proof. The derivation π ends in:

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1 & \vdots\,\pi_2\\ \Gamma \Rightarrow \Delta, \varphi & \varphi, \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

We distinguish cases according to the form of φ.

Suppose φ is atomic. Observe that by the condition that all $\mathrm{Cut}_{\mathrm{CS}}$ inferences in π_1 and π_2 must have rank $< \mathrm{dp}(\varphi) = 0$, there can be no $\mathrm{Cut}_{\mathrm{CS}}$ inferences in π_1 or π_2. We show that $\Gamma \Rightarrow \Delta$ has a cut-free proof by induction on $\mathrm{ht}(\pi_2)$. If $\mathrm{ht}(\pi_2) = 0$, then $\varphi, \Gamma \Rightarrow \Delta$ is an axiom. If φ is not principal, some atomic χ is an element of both Γ and Δ, and $\Gamma \Rightarrow \Delta$ itself is an axiom. If φ is principal, then $\Delta = \Delta, \varphi$. In this case, π_1 ends in $\Gamma \Rightarrow \Delta', \varphi, \varphi$. We obtain a cut-free proof of $\Gamma \Rightarrow \Delta', \varphi$ by applying **??** (admissibility of contraction). If $\mathrm{ht}(\pi_2) > 0$ it ends in a rule R. Take a premise of that rule: it has the form $\varphi, \Gamma', \Pi \Rightarrow \Delta', \Lambda$, where Π and Λ are the active formulas of R. **??** applied to π_1 and π_2 yields derivations of $\Gamma, \Gamma', \Pi \Rightarrow \Delta, \Delta', \Lambda, \varphi$ and $\varphi, \Gamma, \Gamma', \Pi \Rightarrow \Delta, \Delta', \Lambda$, to which the induction hypothesis applies. This gives us a derivation of $\Gamma, \Gamma', \Pi \Rightarrow \Delta, \Delta', \Lambda$, for each premise of R. Applying R yields $\Gamma, \Gamma \Rightarrow \Delta, \Delta$, and **??** provides the cut-free proof of $\Gamma \Rightarrow \Delta$.

If φ is not atomic, we distinguish cases according to the main operator of φ. In each case, we apply Lemma cut.4 to obtain derivations of the premises of the left- and right-rules with φ as principal formula. We then proceed as in case E of the proof of Lemma cut.2.

1. $\varphi \equiv \neg\psi$. The derivations π_1 and π_2 end in $\Gamma \Rightarrow \Delta, \neg\psi$ and $\neg\psi, \Gamma \Rightarrow \Delta$. By Lemma cut.4, there are derivations π_1' and π_2' of $\psi, \Gamma \Rightarrow \Delta$ and

$\Gamma \Rightarrow \Delta, \psi$. The derivation π' is:

$$\dfrac{\begin{array}{cc}\vdots\,\pi'_2 & \vdots\,\pi'_1\\ \Gamma \Rightarrow \Delta, \psi & \psi, \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut_{CS}}$$

2. $\varphi \equiv (\psi \vee \chi)$. The derivations π_1 and π_2 end in $\Gamma \Rightarrow \Delta, \psi \vee \chi$ and $\psi \vee \chi, \Gamma \Rightarrow \Delta$. By Lemma cut.4 (inversion for for $\vee$R applied to π_1), there is a derivation π'_1 of $\Gamma \Rightarrow \Delta, \psi, \chi$. By Lemma cut.4 (inversion for $\vee$L applied to π_2), there is a derivation π'_2 of $\psi, \Gamma \Rightarrow \Delta$ and a derivation π''_2 of $\chi, \Gamma \Rightarrow \Delta$. By applying **??** to π''_2 we also get a derivation π'''_2 of $\chi, \Gamma \Rightarrow \Delta, \psi$. The derivation π' is:

$$\dfrac{\dfrac{\begin{array}{cc}\vdots\,\pi'_1 & \vdots\,\pi'''_2\\ \Gamma \Rightarrow \Delta, \psi, \chi & \chi, \Gamma \Rightarrow \Delta, \psi\end{array}}{\Gamma \Rightarrow \Delta, \psi}\,\mathrm{Cut_{CS}} \qquad \begin{array}{c}\vdots\,\pi'_2\\ \psi, \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut_{CS}}$$

3. $\varphi \equiv (\psi \wedge \chi)$. Exercise.

4. $\varphi \equiv (\psi \to \chi)$. The derivations π_1 and π_2 end in $\Gamma \Rightarrow \Delta, \psi \to \chi$ and $\psi \to \chi, \Gamma \Rightarrow \Delta$. By Lemma cut.4 (inversion for $\to$R applied to π_1), there is a derivation π'_1 of $\psi, \Gamma \Rightarrow \Delta, \chi$. By Lemma cut.4 (inversion for $\to$L applied to π_2), there is a derivation π'_2 of $\Gamma \Rightarrow \Delta, \psi$ and a derivation π''_2 of $\chi, \Gamma \Rightarrow \Delta$. By applying **??** to π''_2 we also get a derivation π'''_2 of $\chi, \psi, \Gamma \Rightarrow \Delta$. The derivation π' is:

$$\dfrac{\begin{array}{cc}\vdots\,\pi'_2 & \dfrac{\begin{array}{cc}\vdots\,\pi'_1 & \vdots\,\pi'''_2\\ \psi, \Gamma \Rightarrow \Delta, \chi & \chi, \psi, \Gamma \Rightarrow \Delta\end{array}}{\psi, \Gamma \Rightarrow \Delta}\,\mathrm{Cut_{CS}}\\ \Gamma \Rightarrow \Delta, \psi & \end{array}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut_{CS}}$$

5. $\varphi \equiv \forall x\, \psi(x)$. The derivations π_1 and π_2 end in $\Gamma \Rightarrow \Delta, \forall x\, \psi(x)$ and $\forall x\, \psi(x), \Gamma \Rightarrow \Delta$. By Lemma cut.4, there is a derivation π'_1 of $\Gamma \Rightarrow \Delta, \psi(c)$.

   Now consider the derivation π_2. It ends in $\forall x\, \psi(x), \Gamma \Rightarrow \Delta$. Moving upward from the end-sequent of π_2, mark every occurrence of $\forall x\, \psi(x)$ on the left of a sequent that "leads to" the occurrence of $\forall x\, \psi(x)$ in the end-sequent of π_2, that is: (a) Mark $\forall x\, \psi(x)$ in the end-sequent of π_2. (b) If $\forall x\, \psi(x)$ occurs in the context of the conclusion of a rule and is marked, mark a corresponding occurrence in the premise(s) of the rule. (c) If $\forall x\, \psi(x)$ is principal in the conclusion of a $\forall$L rule and is marked,

mark the corresponding active occurrences of $\forall x\, \psi(x)$ and $\psi(t)$ in the premise.

Now consider all these marked occurrences of $\forall x\, \psi(x)$. None of them are active in a rule other than $\forall$L (only active formulas of $\forall$L get marked). Delete all marked occurrences of $\forall x\, \psi(x)$ in π_2. If this is a correct derivation, the marked occurrences of $\forall x\, \psi(x)$ were never active; they all came directly from axioms. The derivation ends in $\Gamma \Rightarrow \Delta$ and we can replace π by it. We have a removed a cut of maximal rank.

Otherwise, what we have is not a correct proof because we have deleted formulas $\forall x\, \psi(x)$ which were active in some $\forall$L inferences. We have turned these into "inferences" of the form

$$\dfrac{\begin{array}{c}\vdots\\ \pi_t\end{array}}{\dfrac{\psi(t), \Pi \Rightarrow \Lambda}{\Pi \Rightarrow \Lambda}\,\forall\mathrm{L}}$$

Replace each topmost such "inference" by

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1'[t/c][\Pi \Rightarrow \Lambda]\end{array}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda, \psi(t)} \qquad \dfrac{\begin{array}{c}\vdots\\ \pi_t[\Gamma \Rightarrow \Delta]\end{array}}{\psi(t), \Gamma, \Pi \Rightarrow \Delta, \Lambda}\,\mathrm{Cut_{CS}}$$
$$\Gamma, \Pi \Rightarrow \Delta, \Lambda$$

and add Γ to the left and Δ to the right of every sequent below it. Repeat until all $\forall$L "inferences" with a marked $\psi(t)$ in the premise have been replaced by a $\mathrm{Cut_{CS}}$ inference on some $\psi(t)$. The end-sequent of the resulting derivation (and it now is a correct derivation) is of the form $\Gamma, \ldots, \Gamma \Rightarrow \Delta, \ldots, \Delta$. Apply admissibility of contraction to turn it into a derivation of $\Gamma \Rightarrow \Delta$ and replace π with it. We have replaced one cut on $\forall x\, \psi(x)$ with (possibly many) cuts on $\psi(t)$, but these are all of lower rank. Any cuts already occurring in π_1 or π_2 remain present, but they are also all of lower rank. □

Problem cut.5. Verify the case where $\varphi \equiv (\psi \wedge \chi)$ in the proof of Lemma cut.4. That is, show that if there is a derivation π ending in

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\end{array}}{\Gamma \Rightarrow \Delta, \psi \wedge \chi} \qquad \dfrac{\begin{array}{c}\vdots\\ \pi_2\end{array}}{\psi \wedge \chi, \Gamma \Rightarrow \Delta}\,\mathrm{Cut_{CS}}$$
$$\Gamma \Rightarrow \Delta$$

with π_1 and π_2 containing only cuts of rank $< \mathrm{dp}(\psi \wedge \chi)$ then there is a derivation π' of $\Gamma \Rightarrow \Delta$ containing only cuts of rank $< \mathrm{dp}(\psi \wedge \chi)$.

Lemma cut.2 establishes that we can replace a Cut_{CS} inference of highest rank in a derivation by Cut_{CS} inferences of lower rank. We can again use this lemma to show that we can eliminate any number of Cut_{CS} inferences from a derivation by successively applying it to the topmost sub-derivations ending in maximal Cut_{CS}.

pt:cut:inv: thm:cut-elim-G3c-largest

Theorem cut.6. *Any derivation π in* **G3c** $+ \text{Cut}_{\text{CS}}$ *can be transformed into a proof in* **G3c**.

Proof. First we prove that all cuts of maximal rank in π can be removed. Let d be the maximal rank of cuts occuring in π, and let n be the number of cuts of rank d. The proof is by induction on n. If $n = 0$ there is nothing to prove. If $n > 0$, pick a topmost cut of rank d, and let π_1 be the sub-derivation ending in it. By Lemma cut.5, there is a derivation π_1' of the same end-sequent with all cuts of rank $< d$. Replace π_1 by π_1' in π. This results in a derivation containing $n - 1$ cuts of rank d, and the induction hypothesis applies.

Now the result follows by induction on d. If $d = 0$, all cuts are atomic, and by the previous result can all be removed. If $d > 0$, the previous result yields a proof in which all cuts of rank d have been replaced by cuts of rank $< d$. Since d was the maximum rank of cuts in π, we have proof in which the maximal rank of cuts is $< d$, and the induction hypothesis applies. □

cut.4 The Midsequent Theorem and Herbrand's Theorem

pt:cut:mid: sec

An important consequence of the cut-elimination theorem is the midsequent theorem. It states that if there is a derivation π of $\Gamma \Rightarrow \Delta$ in which every formula is prenex, then there is a derivation π' containing a sequent $S \equiv \Pi \Rightarrow \Lambda$ (called the *midsequent*) such that all inferences above S in π' are propositional and all inferences below S are quantifier inferences. (A formula φ is *prenex* (or in *prenex form*) if it is of the form

$$\mathsf{Q}_1 x_1 \dots \mathsf{Q}_n x_n \, \psi(x_1, \dots, x_n)$$

where each Q_i is either $\forall$ or $\exists$.) An important consequence of the midsequent theorem is Herbrand's theorem. Its general case is a little hard to describe, but very generally it takes the form: for every derivable sentence φ of first-order logic, there is a propositional tautology φ' from which φ can be derived. Here is the version for sentences of the form $\exists x \, \psi(x)$ where ψ is quantifier-free:

Theorem cut.7 (Herbrand's Theorem). *Suppose that $\psi(x)$ is quantifier-free and that $\vdash \exists x \, \psi(x)$. Then there are terms $t_1, \dots, t_n$ such that $\vdash \psi(t_1) \vee \dots \vee \psi(t_n)$.*

The formula $\psi(t_1) \vee \dots \vee \psi(t_n)$ is called a *Herbrand disjunction* of $\exists x \, \psi(x)$. Since ψ is quantifier-free, this disjunction is also quantifier-free. So if it is

derivable, it is derivable by a cut-free proof and consequently with a derivation that only uses propositional inferences. Hence, it is a tautology.

The difficult part of Herbrand's Theorem is showing that a Herbrand disjunction exists for every derivable formula $\exists x\, \psi(x)$. The converse, i.e., that if $\exists x\, \psi(x)$ has a Herbrand disjunction, then it is provable, is easy. In fact, the Herbrand disjunction entails $\exists x\, \psi(x)$. E.g., here is a derivation in **G3c** for the case where $n = 2$:

$$\dfrac{\dfrac{\dfrac{\psi(t_1) \Rightarrow \exists x\, \psi(x), \psi(t_1), \psi(t_2) \qquad \psi(t_2) \Rightarrow \exists x\, \psi(x), \psi(t_1), \psi(t_2)}{\psi(t_1) \vee \psi(t_2) \Rightarrow \exists x\, \psi(x), \psi(t_1), \psi(t_2)}\vee\mathrm{L}}{\psi(t_1) \vee \psi(t_2) \Rightarrow \exists x\, \psi(x), \psi(t_2)}\exists\mathrm{R}}{\psi(t_1) \vee \psi(t_2) \Rightarrow \exists x\, \psi(x)}\exists\mathrm{R}$$

To extract the Herbrand disjunction from a derivation π of $\Rightarrow \exists x\, \psi(x)$, consider the case where all $\exists$R inferences in a cut-free derivation π are at the end:

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\\ \Rightarrow \exists x\, \psi(x), \psi(t_1), \ldots, \psi(t_n)\end{array}}{\begin{array}{c}\Rightarrow \exists x\, \psi(x), \psi(t_2), \ldots, \psi(t_n)\\ \vdots\\ (n-1) \times \exists\mathrm{R}\\ \vdots\\ \Rightarrow \exists x\, \psi(x)\end{array}}\exists\mathrm{R}$$

If these are all the $\exists$R inferences in π with $\exists x\, \psi(x)$ active, then there are no other quantifier inferences in π_1'. That's because $\psi(x)$ contains no other quantifiers, and the end-sequent doesn't contain any formulas on the left. In other words, $\Rightarrow \exists x\, \psi(x), \psi(t_1), \ldots, \psi(t_n)$ is the mid-sequent of π. Moreover, since there are no quantifier inferences above the mid-sequent, all sequents above it contain $\exists x\, \psi(x)$ and these are neither active nor principal. So they can be deleted from the derivation π_1, and we get a derivation of $\Rightarrow \psi(t_1), \ldots, \psi(t_n)$, and using $n - 1$ applications of $\vee$R, a derivation of the Herbrand disjunction $\Rightarrow \psi(t_1) \vee \cdots \vee \psi(t_n)$.

The trouble is that we can't assume that even a cut-free derivation of $\Rightarrow \exists x\, \psi(x)$ has all its $\exists$R inferences in one block at the end. There might be inferences in between these $\exists$R inferences in which some $\psi(t_i)$ is principal. Hence, the need for the midsequent theorem.

Theorem cut.8 (Midsequent Theorem). *If there is a* **G3c***-derivation π of $\Gamma \Rightarrow \Delta$ where all formulas in Γ and Δ are prenex, then there is a derivation π' containing a sequent $\Pi \Rightarrow \Lambda$ such that all inferences below it are quantifier inferences and all inferences above it are propositional inferences.* pt:cut:mid:thm:midsequent

Proof. Define the *order* of a quantifier inference in π as the number of propositional inferences below it, and the order $o(\pi)$ of π as the sum of the orders of all its quantifier inferences. If $o(\pi) = 0$, then no quantifier inference has a

propositional inference below it, and we are done: Since all quantifier inferences have one premise, there is a single topmost quantifier inference. Its premise is the midsequent.

If $o(\pi) > 0$, pick a bottom-most quantifier inference with order > 0. Since it has order > 0, there must be a propositional inference below it. Since it is a bottom-most one, its conclusion can't be the premise of a quantifier inference: that second, lower quantifier inference would also have a propositional inference below it otherwise. We distinguish (many!) cases according to what quantifier inference we have, and what the propositional inference below it is. Because the end-sequent consists of prenex formulas only, the principal formula of the quantifier inference can't be active in the following propositional inference. Otherwise the principal formula of that inference would have a quantifier in the scope of a propositional connective. By the sub-formula property, this would have to be a sub-formula of a formula in the end-sequent. So, the principal formula of the quantifier inference is an element of the context of the lower propositional inference.

Here are two examples:

First, suppose we have a $\forall$R which ends in the right premise of $\to$L. Then the relevant sub-derivation of π ends in the sub-derivation π_1.

$$\dfrac{\begin{array}{c}\vdots\,\pi_2\\ \Gamma' \Rightarrow \Delta', \forall x\,\varphi(x), \psi\end{array} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_3\\ \chi, \Gamma' \Rightarrow \Delta', \varphi(c)\end{array}}{\chi, \Gamma' \Rightarrow \Delta', \forall x\,\varphi(x)}\,\forall\mathrm{R}}{\psi \to \chi, \Gamma' \Rightarrow \Delta', \forall x\,\varphi(x)}\,{\to}\mathrm{L}$$

We can assume that π is regular, and so the eigenvariable c does not occur outside π_3; in particular, it does not occur in $\psi \to \chi$ or in π_2. Now consider the proof π_1':

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_2'\\ \Gamma' \Rightarrow \Delta', \forall x\,\varphi(x), \varphi(c), \psi\end{array} \qquad \begin{array}{c}\vdots\,\pi_3\\ \chi, \Gamma' \Rightarrow \Delta', \forall x\,\varphi(x), \varphi(c)\end{array}}{\psi \to \chi, \Gamma' \Rightarrow \Delta', \forall x\,\varphi(x), \varphi(c)}\,{\to}\mathrm{L}}{\psi \to \chi, \Gamma' \Rightarrow \Delta', \forall x\,\varphi(x), \forall x\,\varphi(x)}\,\forall\mathrm{R}$$

where π_2' is obtained from π_2 by weakening with $\varphi(c)$ on the right. (This does not add any inferences, in particular, no quantifier inferences which might affect the order. It also does not violate any eigenvariable conditions in π_2, since no constants in $\varphi(c)$ are used as eigenvariables in π_2.) The eigenvariable condition on $\forall$R is still satisfied, since c does not occur in $\psi \to \chi$.

We invoke the admissibility of contraction to obtain a derivation π_1'' of $\psi \to \chi, \Gamma' \Rightarrow \Delta', \forall x\,\varphi(x)$. The proof of the admissibility of CR for $\forall x\,\varphi(x)$ and the proof of invertibility of $\forall$R used in that proof did not change the order of any inference: it, at most, deleted inferences. Thus, there are no

more quantifier inferences in π_1'' than in π_1, and each quantifier inference in π_1'' has the same number of proposional inferences below it as its corresponding inference in π_1—with the exception of the last $\forall$R: it had a $\to$L below it in π, which we have moved above it in π_1''. Replace π_1 in π by π_1''. The resulting derivation has a lower order, since at least the order of the $\forall$R inference has been reduced by (at least) 1. Now apply the inductive hypothesis.

The cases where the quantifier inference is $\exists$R is even easier, since contraction is not needed and no eigenvariable condition needs to be worried about. E.g., suppose $\exists$R is followed by $\wedge$R (this time as the left premise). The relevant sub-derivation is π_1:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi_2\end{array}}{\Gamma' \Rightarrow \Delta', \exists x\, \varphi(x), \varphi(t), \psi}}{\Gamma' \Rightarrow \Delta', \exists x\, \varphi(x), \psi}\,\exists\mathrm{R} \qquad \begin{array}{c}\vdots\\ \pi_3\end{array}}{\Gamma' \Rightarrow \Delta', \exists x\, \varphi(x), \psi \wedge \chi}\,\wedge\mathrm{R}$$

Replace π_1 in π by π_1':

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi_2\end{array}}{\Gamma' \Rightarrow \Delta', \exists x\, \varphi(x), \varphi(t), \psi} \qquad \begin{array}{c}\vdots\\ \pi_3'\end{array}}{\dfrac{\Gamma' \Rightarrow \Delta', \exists x\, \varphi(x), \varphi(t), \psi \wedge \chi}{\Gamma' \Rightarrow \Delta', \exists x\, \varphi(x), \psi \wedge \chi}\,\exists\mathrm{R}}\,\wedge\mathrm{R}$$

where π_3' is obtained from π_3 by weakening with $\varphi(t)$ on the right. This weakening involves merely adding $\varphi(t)$ to the right side of every sequent in π_3, so does not change the order of any quantifier inference. The order of quantifier inferences is the same in π' as in π, except the order of the $\exists$R inference permuted with $\wedge$R has decreased by 1. The inductive hypothesis applies. □

Problem cut.6. Describe the cases in the proof of Theorem cut.8 where $\exists$L ends in the premise of $\to$R, and where $\forall$L ends in the left premise of $\vee$L.

Proof of Herbrand's Theorem. Suppose π ends in $\Rightarrow \exists x\, \varphi(x)$. By Theorem cut.8, we can transform π into a derivation π' with a midsequent S, which must be of the form

$$\Rightarrow \exists x\, \varphi(x), \varphi(t_1), \ldots, \varphi(t_n).$$

Since all inferences above S are propositional, $\exists x\, \varphi(x)$ cannot be principal in an inference above S. Since it is not atomic, it cannot be principal in an axiom. Since $\exists x\, \varphi(x)$ cannot be a proper subformula of $\varphi(t_1)$ (remember that $\varphi(x)$ is quantifier-free), it can also not be active above S. Deleting $\exists x\, \varphi(x)$ from the sub-derivation ending in S therefore yields a correct derivation of $\Rightarrow \varphi(t_1), \ldots, \varphi(t_n)$, from which a derivation of $\Rightarrow \varphi(t_1) \vee \cdots \vee \varphi(t_n)$ can be obtained by $n - 1$ $\vee$R inferences. □

cut.5 Maehara's Lemma and Craig's Interpolation Theorem

pt:cut:itp: sec

The interpolation theorem, due to William Craig, states that if $\varphi \vDash \psi$ then there is a sentence χ (the *interpolant*) such that $\varphi \vDash \chi$ and $\chi \vDash \psi$. Importantly, the interpolant χ can be chosen so that the constant symbols and predicate symbols occurring in χ all occur in both φ and ψ. (We assume that no function symbols are involved.) Without this restriction, φ and ψ could trivially serve as interpolants.

The interpolation theorem holds for classical first-order logic and has been called the "last important property of first-order logic" to be discovered. It has important applications in model theory, and automated reasoning. Its original motivation and application was in the philosophy of science, since it can be used to study the primitives by which a theory can or cannot be axiomatized. It also implies Beth's definability theorem, which states that if a theory can implicitly define a concept, it can define it explicitly.

Like many results in mathematical logic, the interpolation theorem can be proved both using the methods of model theory and those of proof theory. The advantage of proof theoretic methods is that not only do they show the existence of an interpolant, but provide an algorithm that computes the interpolant from a derivation that $\varphi \vDash \psi$, e.g., a derivation in **G3c** of $\varphi \Rightarrow \psi$.

Let $\mathcal{L}(\varphi)$ be the set of constant symbols and predicate symbols in φ and $\mathcal{L}(\Gamma) = \bigcup\{\mathcal{L}(\varphi) : \varphi \in \Gamma\}$. Throughout this section, we assume we are dealing with formulas without function symbols.

pt:cut:itp: thm:interpolation

Theorem cut.9 (Craig's Interpolation Theorem). *Let the language of φ be $\mathcal{L}(\varphi)$ and that of ψ be $\mathcal{L}(\psi)$. If $\mathbf{G3c} \vdash \varphi \Rightarrow \psi$ then there is a formula χ in the language $\mathcal{L}(\chi) \subseteq \mathcal{L}(\varphi) \cap \mathcal{L}(\psi)$ such that $\mathbf{G3c} \vdash \varphi \Rightarrow \chi$ and $\mathbf{G3c} \vdash \chi \Rightarrow \psi$.*

We prove a generalization of the interpolation theorem to make use of the structure of derivations in **G3c**. To this end, imagine the antecendent and succedents of sequents to be separated into two parts each. We write $\Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2$ to indicate that Γ_1 and Δ_1 belong to the first part and Γ_2 and Δ_2 to the second part. The interpolation theorem then follows immediately from the following lemma.

pt:cut:itp: lem:maehara

Lemma cut.10 (Maehara's Lemma). *If $\mathbf{G3c} \vdash \Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2$ then there is a formula χ in the language $\mathcal{L}(\chi) \subseteq \mathcal{L}(\Gamma_1, \Delta_1) \cap \mathcal{L}(\Gamma_2, \Delta_2)$ such that $\mathbf{G3c} \vdash \Gamma_1 \Rightarrow \Delta_1, \chi$ and $\mathbf{G3c} \vdash \chi, \Gamma_2 \Rightarrow \Delta_2$.*

Proof. Assume that $\pi \vdash \Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2$. We prove the claim by induction on $n = \mathrm{ht}(\pi)$.

If $n = 0$, then $\Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2$ is an axiom, and some atomic formula φ occurs in both Γ_1, Γ_2 and Δ_1, Δ_2. There are four cases depending on whether φ belongs to the first part or the second on the left, and whether it belongs to the first or second part on the right.

1. $\varphi \in \Gamma_1$ and $\varphi \in \Delta_1$. We want a χ such that both $\Gamma_1 \Rightarrow \Delta_1, \chi$ and $\chi, \Gamma_2 \Rightarrow \Delta_2$ are derivable. The first is already an axiom regardless of how we pick χ since φ occurs on both the left and right. Our only option to guarantee that $\chi, \Gamma_2 \Rightarrow \Delta_2$ is derivable is to let $\chi \equiv \bot$.

2. $\varphi \in \Gamma_1$ and $\varphi \in \Delta_2$. Then $\Gamma_1 \Rightarrow \Delta_1, \varphi$ and $\varphi, \Gamma_2 \Rightarrow \Delta_2$ are both axioms, so we can pick $\chi \equiv \varphi$.

3. $\varphi \in \Gamma_2$ and $\varphi \in \Delta_1$. Then $\varphi, \Gamma_1 \Rightarrow \Delta_1$ and $\Gamma_2 \Rightarrow \Delta_2, \varphi$ would be axioms, but they have φ on the wrong sides. So φ isn't the interpolant. However, using $\neg$R and $\neg$L we can derive $\Gamma_1 \Rightarrow \Delta_1, \neg\varphi$ and $\neg\varphi, \Gamma_2 \Rightarrow \Delta_2$.

4. $\varphi \in \Gamma_2$ and $\varphi \in \Delta_2$. Exercise.

The cases where $\Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2$ is an axiom because $\bot \in \Gamma_1$ or $\bot \in \Gamma_2$ are left as exercises.

If $n > 0$, π ends in a logical inference. We have to distinguish cases cases according to which rule was used, and to which part the principal formula belongs. In each case we can use the induction hypothesis that the lemma holds for all derivations π_1 with $\mathrm{ht}(\pi_1) < n$, and however the end-sequent of π_1 may be partitioned into two parts. Let's consider some specific cases.

1. π ends in $\neg$R with principal formula $\neg\varphi$. We have two cases: depending on whether $\neg\varphi$ belongs to the first or second part, π ends in either

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \varphi, \Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2\end{array}}{\Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1, \neg\varphi \,;\, \Delta_2}\,\neg\mathrm{R}$$

   or

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \Gamma_1 \,;\, \varphi, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2\end{array}}{\Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2, \neg\varphi}\,\neg\mathrm{R}$$

   Note that if the principal formula $\neg\varphi$ belongs to the first part we assigned the active formula φ also to the first part the premise. This is a choice we get to make. The inductive hypothesis would also apply if we assigned the active formula to the other part, but then we would not be able to find an interpolant (exercise: try it).

   Consider the first situation first, where $\neg\varphi$ belongs to the first part. By inductive hypothesis, there is an interpolant of the end-sequent of π_1, i.e., a formula χ_1 such that both

$$\varphi, \Gamma_1 \Rightarrow \Delta_1, \chi_1 \qquad \text{and} \qquad \chi_1, \Gamma_2 \Rightarrow \Delta_2$$

are provable. What we need is a formula χ such that both of the following are derivable:

$$\Gamma_1 \Rightarrow \Delta_1, \neg\varphi, \chi \qquad \text{and} \qquad \chi, \Gamma_2 \Rightarrow \Delta_2$$

Clearly, we can just take $\chi \equiv \chi_1$: the inductive hypothesis already told us that the sequent on the right is derivable, and the sequent on the left follows from the sequent we got from the inductive hypothesis by $\neg$R. The inductive hypothesis also tells us that $\mathcal{L}(\chi_1) \subseteq \mathcal{L}(\varphi, \Gamma_1, \Delta_1) \cap \mathcal{L}(\Gamma_2, \Delta_2)$. Since $\mathcal{L}(\neg\varphi) = \mathcal{L}(\varphi)$ and $\mathcal{L}(\chi) = \mathcal{L}(\chi_1)$ we have that $\mathcal{L}(\chi_1) \subseteq \mathcal{L}(\Gamma_1, \Delta_1, \neg\varphi) \cap \mathcal{L}(\Gamma_2, \Delta_2)$.

In the second scenario the situation is slightly different in that $\neg\varphi$ belongs to the second part. We assign the active formula in the premise also to the second part and apply the inductive hypothesis to

$$\Gamma_1 \Rightarrow \Delta_1, \chi_1 \qquad \text{and} \qquad \chi_1, \varphi, \Gamma_2 \Rightarrow \Delta_2$$

are provable. If we again apply the $\neg$R rule to the second of these, we get derivations of

$$\Gamma_1 \Rightarrow \Delta_1, \chi_1 \qquad \text{and} \qquad \chi_1, \Gamma_2 \Rightarrow \Delta_2, \neg\varphi$$

These are again exactly the sequents we want to be derivable if χ_1 is an interpolant; we again let $\chi \equiv \chi_1$. $\mathcal{L}(\chi_1) \subseteq \mathcal{L}(\Gamma_1, \Delta_1) \cap \mathcal{L}(\Gamma_2, \Delta_2, \neg\varphi)$ as before.

2. π ends in $\to$R with principal formula $\varphi \to \psi$. We again have two cases, depending on whether $\varphi \to \psi$ belongs to the first or second part. The derivation π ends in either

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\end{array}}{\varphi, \Gamma_1 \,;\, \Gamma_2 \;\Rightarrow\; \Delta_1, \psi \,;\, \Delta_2} \to\!\text{R}$$
$$\Gamma_1 \,;\, \Gamma_2 \;\Rightarrow\; \Delta_1, \varphi \to \psi \,;\, \Delta_2$$

or

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\\ \vdots\end{array}}{\Gamma_1 \,;\, \varphi, \Gamma_2 \;\Rightarrow\; \Delta_1 \,;\, \Delta_2, \psi} \to\!\text{R}$$
$$\Gamma_1 \,;\, \Gamma_2 \;\Rightarrow\; \Delta_1 \,;\, \Delta_2, \varphi \to \psi$$

Again we have assigned the active formulas to the same part in the premise as the principal formula in the conclusion. In the first secenario, we apply the inductive hypothesis and get derivations of

$$\varphi, \Gamma_1 \Rightarrow \Delta_1, \psi, \chi_1 \qquad \text{and} \qquad \chi_1, \Gamma_2 \Rightarrow \Delta_2$$

The sequent on the left is a suitable premise for →R. So we have the following derivable sequents:

$$\Gamma_1 \Rightarrow \Delta_1, \varphi \to \psi, \chi_1 \qquad \text{and} \qquad \chi_1, \Gamma_2 \Rightarrow \Delta_2$$

This means that χ_1 is also an interpolant for the end-sequent of π and we can take $\chi \equiv \chi_1$ once more.

The other scenario is handled the same way.

3. π ends in →L with principal formula $\varphi \to \psi$. If $\varphi \to \psi$ belongs to the first part, the derivation π ends in

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1 & \vdots\,\pi_2 \\ \Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1, \varphi \,;\, \Delta_2 & \psi, \Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2\end{array}}{\varphi \to \psi, \Gamma_1 \,;\, \Gamma_2 \Rightarrow \Delta_1 \,;\, \Delta_2}$$

The inductive hypothesis now gives us four derivable sequents, two for the interpolant χ_1 of the left premise and two for the interpolant χ_2 of the second premise:

$$\begin{array}{llll} \Gamma_1 \Rightarrow \Delta_1, \varphi, \chi_1 & (1) & \chi_1, \Gamma_2 \Rightarrow \Delta_2 & (2) \\ \psi, \Gamma_1 \Rightarrow \Delta_1, \chi_2 & (3) & \chi_2, \Gamma_2 \Rightarrow \Delta_2 & (4) \end{array}$$

Sequents (1) and (3) can serve as premises of →L, if we apply weakening and add χ_2 on the right to (1) and χ_1 on the right to (3):

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ \Gamma_1 \Rightarrow \Delta_1, \varphi, \chi_1, \chi_2 & \psi, \Gamma_1 \Rightarrow \Delta_1, \chi_1, \chi_2\end{array}}{\varphi \to \psi, \Gamma_1 \Rightarrow \Delta_1, \chi_1, \chi_2}\,{\to}\mathrm{L}$$

By applying ∨R we get the sequent

$$\varphi \to \psi, \Gamma_1 \Rightarrow \Delta_1, \chi_1 \vee \chi_2$$

This suggests that $\chi_1 \vee \chi_2$ might be our interpolant in this case. And indeed, we can derive the other required sequent from the remaining sequents (2) and (4):

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ \chi_1, \Gamma_2 \Rightarrow \Delta_2 & \chi_2, \Gamma_2 \Rightarrow \Delta_2\end{array}}{\chi_1 \vee \chi_2, \Gamma_2 \Rightarrow \Delta_2}\,{\vee}\mathrm{L}$$

It remains to verify that $\chi_1 \vee \chi_2$ is in the correct language. The inductive hypothesis gives us that

$$\begin{array}{ll} \mathcal{L}(\chi_1) \subseteq \mathcal{L}(\Gamma_1, \Delta_1, \varphi) \cap \mathcal{L}(\Gamma_2, \Delta_2) & \text{and} \\ \mathcal{L}(\chi_2) \subseteq \mathcal{L}(\psi, \Gamma_1, \Delta_1) \cap \mathcal{L}(\Gamma_2, \Delta_2) & \end{array}$$

Since

$$\begin{aligned}\mathcal{L}(\varphi) &\subseteq \mathcal{L}(\varphi \to \psi) \qquad \text{and}\\ \mathcal{L}(\psi) &\subseteq \mathcal{L}(\varphi \to \psi)\end{aligned}$$

we have both

$$\begin{aligned}\mathcal{L}(\chi_1) &\subseteq \mathcal{L}(\Gamma_1, \Delta_1, \varphi \to \psi) \cap \mathcal{L}(\Gamma_2, \Delta_2) \qquad \text{and}\\ \mathcal{L}(\chi_2) &\subseteq \mathcal{L}(\Gamma_1, \Delta_1, \varphi \to \psi) \cap \mathcal{L}(\Gamma_2, \Delta_2)\end{aligned}$$

and consequently, $\mathcal{L}(\chi_1 \vee \chi_2) =$

$$\mathcal{L}(\chi_1) \cup \mathcal{L}(\chi_2) \subseteq \mathcal{L}(\Gamma_1, \Delta_1, \varphi \to \psi) \cap \mathcal{L}(\Gamma_2, \Delta_2),$$

which is as required.

In the case where $\varphi \to \psi$ belongs to the second part, the situation is different, but handled in a similar way. The inductive hypothesis again gives us four derivable sequents:

$$\begin{array}{llll}\Gamma_1 \Rightarrow \Delta_1, \chi_1 & (1) & \chi_1, \Gamma_2 \Rightarrow \Delta_2, \varphi & (2)\\ \Gamma_1 \Rightarrow \Delta_1, \chi_2 & (3) & \chi_2, \psi, \Gamma_2 \Rightarrow \Delta_2 & (4)\end{array}$$

Now sequents (2) and (4)—plus some weakened formulas—serve as premises of →L:

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ \chi_1, \chi_2, \Gamma_2 \Rightarrow \Delta_2, \varphi & \psi, \chi_1, \chi_2, \Gamma_2 \Rightarrow \Delta_2\end{array}}{\varphi \to \psi, \chi_1, \chi_2, \Gamma_2 \Rightarrow \Delta_2} \to\text{L}$$

We again want to turn this into a sequent that has a single formula built from χ_1 and χ_2, but this time we need it in the antecedent (since we're now dealing with the second part). Applying ∧L does the trick; we should take $\chi \equiv (\chi_1 \wedge \chi_2)$ as the interpolant. Fortuitously, we can use the remaining sequents (1) and (3) to derive the corresponding sequent for the other part:

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ \Gamma_1 \Rightarrow \Delta_1, \chi_1 & \Gamma_1 \Rightarrow \Delta_1, \chi_2\end{array}}{\Gamma_1 \Rightarrow \Delta_1, \chi_1 \wedge \chi_2} \wedge\text{R}$$

As before, we have $\mathcal{L}(\chi_1 \wedge \chi_2) \subseteq \mathcal{L}(\Gamma_1, \Delta_1) \cap \mathcal{L}(\Gamma_2, \Delta_2, \varphi \to \psi)$.

4. π ends in $\forall$R with principal formula $\forall x\,\varphi(x)$:

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdots\end{array}}{\Gamma_1\,;\,\Gamma_2 \Rightarrow \Delta_1, \varphi(c)\,;\,\Delta_2}\,\forall\mathrm{R}$$
$$\dfrac{\Gamma_1\,;\,\Gamma_2 \Rightarrow \Delta_1, \forall x\,\varphi(x)\,;\,\Delta_2}$$

or

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdots\end{array}}{\Gamma_1\,;\,\Gamma_2 \Rightarrow \Delta_1\,;\,\Delta_2, \varphi(c)}\,\forall\mathrm{R}$$
$$\dfrac{\Gamma_1\,;\,\Gamma_2 \Rightarrow \Delta_1\,;\,\Delta_2, \forall x\,\varphi(x)}$$

In the first scenario, the induction hypothesis gives us derivations of $\Gamma_1 \Rightarrow \Delta_1, \varphi(c), \chi_1$ and $\chi_1, \Gamma_2 \Rightarrow \Delta_2$. We can apply $\forall$R to the first to obtain $\Gamma_1 \Rightarrow \Delta_1, \forall x\,\varphi(x), \chi_1$. (The eigenvariable condition is satisfied, since it was already satisfied in the $\forall$R rule at the end of π.) Thus χ_1 would be an interpolant if the condition that $\mathcal{L}(\chi_1) \subseteq \mathcal{L}(\Gamma_1, \Delta_1, \forall x\,\varphi(x)) \cap \mathcal{L}(\Gamma_2, \Delta_2)$ is satisfied. By the inductive hypothesis, we have $\mathcal{L}(\chi_1) \subseteq \mathcal{L}(\Gamma_1, \Delta_1, \varphi(c)) \cap \mathcal{L}(\Gamma_2, \Delta_2)$. So we have to worry about c—can c occur in χ_1? It can't. If c occurs in χ_1, we would have both $c \in \mathcal{L}(\Gamma_1, \Delta_1, \varphi(c))$ (which is true) and also $c \in \mathcal{L}(\Gamma_2, \Delta_2)$. But then c would occur in the conclusion of the $\forall$R inference in π, violating the eigenvariable condition.

The other scenario is similar.

5. π ends in $\exists$R with principal formula $\exists x\,\varphi(x)$. We again have two scenarios. We consider the scenario where $\exists x\,\varphi(x)$ belongs to the first part, and leave the other as an exercise.

In the first scenario, π ends in

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdots\end{array}}{\Gamma_1\,;\,\Gamma_2 \Rightarrow \Delta_1, \exists x\,\varphi(x), \varphi(t)\,;\,\Delta_2}\,\exists\mathrm{R}$$
$$\dfrac{\Gamma_1\,;\,\Gamma_2 \Rightarrow \Delta_1, \exists x\,\varphi(x)\,;\,\Delta_2}$$

The induction hypothesis gives us derivations of $\Gamma_1 \Rightarrow \Delta_1, \exists x\,\varphi(x), \varphi(t), \chi_1$ and $\chi_1, \Gamma_2 \Rightarrow \Delta_2$. We can apply $\exists$R to the first to obtain $\Gamma_1 \Rightarrow \Delta_1, \exists x\,\varphi(x), \chi_1$. Thus χ_1 would again be an interpolant if the condition that $\mathcal{L}(\chi_1) \subseteq \mathcal{L}(\Gamma_1, \Delta_1, \exists x\,\varphi(x)) \cap \mathcal{L}(\Gamma_2, \Delta_2)$ is satisfied. Here the inductive hypothesis gives us $\mathcal{L}(\chi_1) \subseteq \mathcal{L}(\Gamma_1, \Delta_1, \exists x\,\varphi(x), \varphi(t)) \cap \mathcal{L}(\Gamma_2, \Delta_2)$. Now remember that we assumed that there are no function symbols. That means the term t can only be a constant symbol, i.e., $t \equiv b$. If $b \notin \mathcal{L}(\chi_1)$ or $b \in \mathcal{L}(\Gamma_1, \Delta_1, \exists x\,\varphi(x)) \cap \mathcal{L}(\Gamma_2, \Delta_2)$ there is nothing to worry about: we can take χ_1 as the interpolant. But what if $b \in \mathcal{L}(\chi_1)$ and $b \notin \mathcal{L}(\Gamma_1, \Delta_1, \exists x\,\varphi(x)) \cap \mathcal{L}(\Gamma_2, \Delta_2)$? First of all, by replacing all occurrences

of b in χ_1 by a variable y we can write $\chi_1 \equiv \chi_1'[b/y]$ where χ_1' does not contain b. Furthermore, either $b \notin \mathcal{L}(\Gamma_1, \Delta_1, \exists x\, \varphi(x))$ or $b \notin \mathcal{L}(\Gamma_2, \Delta_2)$. We can take $\chi \equiv \forall y\, \chi_1'(y)$ or $\chi \equiv \exists y\, \chi_1'(y)$, respectively. In the first case, we have:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \Rightarrow \Delta_1, \exists x\, \varphi(x), \varphi(b), \chi_1'(b)\end{array}}{\Gamma_1 \Rightarrow \Delta_1, \exists x\, \varphi(x), \chi_1'(b)}\exists\mathrm{R}}{\Gamma_1 \Rightarrow \Delta_1, \exists x\, \varphi(x), \forall y\, \chi_1'(y)}\forall\mathrm{R}$$

and

$$\dfrac{\begin{array}{c}\vdots\\ \chi_1'(b), \forall y\, \chi_1'(y), \Gamma_2 \Rightarrow \Delta_2\end{array}}{\forall y\, \chi_1'(y), \Gamma_2 \Rightarrow \Delta_2}\forall\mathrm{L}$$

The derivations of the top sequents are from the inductive hypothesis (in the second one, we apply admissibility of weakening with $\forall y\, \chi_1'(y)$ in the antecedent). The eigenvariable condition on $\forall$R in the first derivation is satisfied because $b \notin \mathcal{L}(\Gamma_1, \Delta_1, \exists x\, \varphi(x))$, and χ_1' was defined so it does not contain b.

In the second case, we have (from the inductive hypothesis plus admissibility of weakening by $\exists y\, \chi_1'(y)$):

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \Rightarrow \Delta_1, \exists x\, \varphi(x), \varphi(b), \exists y\, \chi_1'(y), \chi_1'(b)\end{array}}{\Gamma_1 \Rightarrow \Delta_1, \exists x\, \varphi(x), \exists y\, \chi_1'(y), \chi_1'(b)}\exists\mathrm{R}}{\Gamma_1 \Rightarrow \Delta_1, \exists x\, \varphi(x), \exists y\, \chi_1'(y)}\exists\mathrm{R}$$

and

$$\dfrac{\begin{array}{c}\vdots\\ \chi_1'(b), \Gamma_2 \Rightarrow \Delta_2\end{array}}{\exists y\, \chi_1'(y), \Gamma_2 \Rightarrow \Delta_2}\exists\mathrm{L}$$

The eigenvariable condition on $\exists$L in the second derivation is satisfied because $b \notin \mathcal{L}(\Gamma_2, \Delta_2)$, and χ_1' was defined so it does not contain b.

The other cases are similary; we leave them as exercises. □

Problem cut.7. Suppose we had a connective $\uparrow$ with rules

$$\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\varphi \uparrow \psi, \Gamma \Rightarrow \Delta}\uparrow\mathrm{L} \qquad \dfrac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi \uparrow \psi}\uparrow\mathrm{R}$$

Show that Maehara's Lemma still holds by carrying out the cases in the proof of Lemma cut.10 where π ends in one of these rules.

Having established Maehara's Lemma, we can use it to prove Craig's Interpolation Theorem.

Proof of Theorem cut.9. Suppose $\varphi \Rightarrow \psi$ is derivable. Apply Lemma cut.10 to $\varphi\ ;\ \Rightarrow\ ;\psi$ to obtain the interpolant χ. We have $\vdash \varphi \Rightarrow \chi$ and $\vdash \chi \Rightarrow \psi$. □

Suppose Γ is a theory (set of sentences) in a language $\mathcal{L}$ containing an n-place predicate R. We say that Γ *implicitly defines* R, iff

$$\Gamma, \Gamma' \vDash \forall x_1 \ldots \forall x_n \left(R(x_1, \ldots, x_n) \leftrightarrow R'(x_1, \ldots, x_n)\right),$$

where Γ' is Γ with all occurrences of R replaced by $R' \notin \mathcal{L}$. Γ *explicitly defines* R iff there is some $\varphi(x_1, \ldots, x_n)$ not containing R such that

$$\Gamma \vDash \forall x_1 \ldots \forall x_n \left(R(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n)\right).$$

Clearly, if Γ explicitly defines R it implicitly defines R. The converse is not obvious, but nevertheless holds:

Lemma cut.11 (Beth Definability Theorem). *If Γ implicitly defines R then it also explicitly defines it.*

Proof. Suppose Γ implicitly defines R. Let R', Γ' be as above and $\mathcal{L}' = \mathcal{L}(\Gamma') = \mathcal{L} \setminus \{R\} \cup \{R'\}$. By hypothesis, we have

$$\Gamma, \Gamma' \Rightarrow \forall x_1 \ldots \forall x_n \left(R(x_1, \ldots, x_n) \leftrightarrow R'(x_1, \ldots, x_n)\right)$$

Let $c_1, \ldots, c_n$ be constant symbols not occurring in Γ. By the invertibility lemma we have

$$\Gamma, \Gamma', R(c_1, \ldots, c_n) \Rightarrow R'(c_1, \ldots, c_n)$$

Apply Maehara's Lemma to

$$\Gamma, R(c_1, \ldots, c_n)\ ;\ \Gamma' \Rightarrow\ ;\ R'(c_1, \ldots, c_n)$$

to find an $\varphi \equiv \varphi(x_1, \ldots, x_n)$ such that

$$\begin{aligned} \Gamma, R(c_1, \ldots, c_n) &\Rightarrow \varphi(c_1, \ldots, c_n) \\ \varphi(c_1, \ldots, c_n), \Gamma' &\Rightarrow R'(c_1, \ldots, c_n) \end{aligned}$$

have derivations. Since $\mathcal{L}(\varphi) = \mathcal{L}_1 \cap \mathcal{L}_2 = \mathcal{L}(\Gamma) \setminus \{R\}$, φ does not contain R. Replace R' everywhere by R in the derivation of the second sequent to get a derivation of

$$\varphi(c_1, \ldots, c_n), \Gamma \Rightarrow R(c_1, \ldots, c_n)$$

Apply →R and ∀R to get a derivation of

$$\Gamma \Rightarrow \forall x_1 \ldots \forall x_n \, (R(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n)).$$

This shows that $\varphi(x_1, \ldots, x_n)$ explicitly defines ρ in Γ. □

A third result which is equivalent to interpolation (and, like it, can be proved using Maehara's Lemma) is the following theorem: If two consistent theories Γ_1, Γ_2 contain a complete theory Γ in the language $\mathcal{L}(\Gamma) \subseteq \mathcal{L}(\Gamma_1) \cap \mathcal{L}(\Gamma_2)$, then $\Gamma_1 \cup \Gamma_2$ is consistent.

Recall that a complete theory is one such that for every sentence φ in its language either $\Gamma \vdash \varphi$ or $\Gamma \vdash \neg\varphi$. Since Γ_1 and Γ_2 are consistent extensions of Γ, Γ must be consistent. It follows that if φ is a sentence in the language $\mathcal{L}(\Gamma_1) \cap \mathcal{L}(\Gamma_2)$, we can't have $\Gamma_1 \models \varphi$ and $\Gamma_2 \models \neg\varphi$. To see this, suppose there were such an φ. If $\Gamma_1 \models \varphi$ then $\Gamma \models \varphi$. Otherwise, by completeness of Γ, $\Gamma \models \neg\varphi$, and since $\Gamma \subseteq \Gamma_1$, $\Gamma_1 \models \neg\varphi$, making Γ_1 inconsistent. So $\Gamma \models \varphi$ and since $\Gamma \subseteq \Gamma_2$, $\Gamma_2 \models \varphi$. So $\Gamma \not\models \neg\varphi$, or Γ_2 would be inconsistent.

The non-existence of a sentence φ in the joint language $\mathcal{L}(\Gamma_1) \cap \mathcal{L}(\Gamma_2)$ with $\Gamma_1 \models \varphi$ and $\Gamma_2 \models \neg\varphi$ is all that's needed for the proof, which we give here proof-theoretically using Maehara's Lemma.

Theorem cut.12 (Robinson's Joint Consistency Theorem). *Suppose Γ_1, Γ_2 are consistent theories and for no φ in the language $\mathcal{L}(\Gamma_1) \cap \mathcal{L}(\Gamma_2)$ do we have $\Gamma_1 \models \varphi$ and $\Gamma_2 \models \neg\varphi$. Then $\Gamma_1 \cup \Gamma_2$ is consistent.*

Proof. Suppose for the contrary that $\Gamma_1 \cup \Gamma_2$ is inconsistent. Then the sequent $\Gamma_1, \Gamma_2 \Rightarrow$ has a derivation. Apply Maehara's Lemma to $\Gamma_1 \,;\, \Gamma_2 \Rightarrow \;;$ to get a sentence φ in the language $\mathcal{L}(\Gamma_1) \cap \mathcal{L}(\Gamma_2)$ such that $\vdash \Gamma_1 \Rightarrow \varphi$ and $\vdash \varphi, \Gamma_2 \Rightarrow$. From the latter we get $\vdash \Gamma_2 \Rightarrow \neg\varphi$. But we assumed that no such sentence exists. □

Above we have tacitly assumed that the theories Γ, Γ_1, Γ_2 are finite. By compactness, the results also hold for infinite theories.

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Bibliography