cut.1 Removing Topmost Cuts

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Definition cut.1. Consider a derivation π ending in a $\mathrm{Cut}_{\mathrm{CS}}$ rule,

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1 & \vdots\,\pi_2\\ \Gamma \Rightarrow \Delta,\varphi & \varphi,\Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

and containing no other $\mathrm{Cut}_{\mathrm{CS}}$ inferences. The *cut height* of π is $\mathrm{ch}(\pi) = \mathrm{ht}(\pi_1) + \mathrm{ht}(\pi_2)$. The *cut rank* of π is $\mathrm{cr}(\pi) = \mathrm{dp}(\varphi)$.

pt:cut:top:lem:cut-adm-G3c

Lemma cut.2. *The* $\mathrm{Cut}_{\mathrm{CS}}$ *rule is admissible in* **G3c**. *In other words, if* π *is a derivation ending in a single* $\mathrm{Cut}_{\mathrm{CS}}$ *and otherwise using only rules of* **G3c**, *then there is a derivation* π' *of the same end-sequent in* **G3c**.

Proof. The proof is by double induction on cut height and rank. The derivation π ends in:

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1 & \vdots\,\pi_2\\ \Gamma \Rightarrow \Delta,\varphi & \varphi,\Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

Induction basis: $\mathrm{ch}(\pi) = 0$ and $\mathrm{cr}(\pi) = 0$. Since $\mathrm{ch}(\pi) = \mathrm{ht}(\pi_1) + \mathrm{ht}(\pi_2) = 0$, both $\Gamma \Rightarrow \Delta,\varphi$ and $\varphi,\Gamma \Rightarrow \Delta$ are axioms. There are two cases: (a) either φ is not principal in one of them, or (b) φ is principal in both of them. In both cases we'll show below that $\Gamma \Rightarrow \Delta$ must be an axiom (and we don't need the inductive hypothesis to do that).

We'll now distinguish a number of possible cases according to whether π_1 and π_2 are axioms or end in an inference, and whether φ is principal in that inference or not. Cases A and B cover the induction basis. In the inductive step we assume $\mathrm{ch}(\pi) > 0$ or $\mathrm{cr}(\pi) > 0$ (or both). We can use the inductive hypothesis, namely that the last $\mathrm{Cut}_{\mathrm{CS}}$ can be removed from any derivation which ends in a single $\mathrm{Cut}_{\mathrm{CS}}$ which either has cut rank $= \mathrm{cr}(\pi)$ and a cut height $< \mathrm{ch}(\pi)$, or has cut rank $< \mathrm{cr}(\pi)$. The case where $\mathrm{ch}(\pi) = 0$ (both premises are axioms) are covered by cases A and B. The cases where $\mathrm{ch}(\pi) > 0$ are covered by cases C–D.

Case A: Some ψ occurs in both Γ and Δ. Then $\Gamma \Rightarrow \Delta$ is an axiom (if ψ is atomic), or has a derivation π' in **G3c** without $\mathrm{Cut}_{\mathrm{CS}}$ by **??**. This covers case (a) of the induction basis, since if φ is not principal in $\Gamma \Rightarrow \Delta,\varphi$ or not principal in $\varphi,\Gamma \Rightarrow \Delta$ and these are axioms, then some other atomic ψ must occur in both Γ and Δ. It also covers the inductive step if one of the premises is an axiom in which φ is not principal (even if the other premise is the conclusion of a rule).

Case B: Both premises are axioms, and φ is principal and hence atomic. Consider the left premise, $\Gamma \Rightarrow \Delta, \varphi$. Since φ is principal, it must occur in Γ (otherwise $\Gamma \Rightarrow \Delta, \varphi$ wouldn't be an axiom). Similarly, φ must occur in Δ, or else $\varphi, \Gamma \Rightarrow \Delta$ wouldn't be an axiom. Thus φ is atomic and occurs in both Γ and Δ, i.e., $\Gamma \Rightarrow \Delta$ is an axiom. This covers case (b) of the induction basis.

Alternative cases A and B

Case A: One of the premises of the cut is an axiom of the form $\chi, \Gamma' \Rightarrow \Delta', \chi$ with χ atomic. If χ is not the cut formula φ, then χ must occur in both Γ and Δ. In that case, $\Gamma \Rightarrow \Delta$ is an axiom and we can take it as π'. If $\chi \equiv \varphi$ is the cut formula then either (if the left premise is an axiom) $\varphi \in \Gamma$ and $\Gamma = \Gamma', \varphi$, or (if the right premise is an axiom) $\varphi \in \Delta$ and $\Delta = \Delta', \varphi$. In the first case, π_2 is a derivation of $\varphi, \varphi, \Gamma' \Rightarrow \Delta$ and by admissibility of CL (**??**) there is a derivation of $\Gamma \Rightarrow \Delta$. In the second case, π_2 is a derivation of $\Gamma \Rightarrow \Delta', \varphi, \varphi$ and by admissibility of CR we get a derivation of $\Gamma \Rightarrow \Delta$.

Case B: One of the premises is an axiom of the form $\bot, \Gamma' \Rightarrow \Delta'$. If the left premise is such an axiom, then $\bot \in \Gamma$ and $\Gamma \Rightarrow \Delta$ is also an axiom. If the right premise is such an axiom, then either $\bot \in \Gamma$ (in which case $\Gamma \Rightarrow \Delta$ again is an axiom), or $\varphi \equiv \bot$. In the latter case, π_1 is a derivation of $\Gamma \Rightarrow \Delta, \bot$. We can turn it into a derivation of $\Gamma \Rightarrow \Delta$ by deleting one copy of $\bot$ from the consequent of every sequent in π_1. (Exercise: verify this by induction on $\mathrm{ht}(\pi_1)$.)

Now suppose that neither case A nor case B applies. In the remaining cases, $\mathrm{ch}(\pi) > 0$, i.e., at least one of the premises is the conclusion of an inference.

Cases C and D: Permuting cuts. The possibilities considered in cases C and D all have a common pattern. We begin with a derivation ending in $\mathrm{Cut_{CS}}$ where one premise is obtained by a rule R in which the cut formula φ is not principal (some other formula θ is). In this derivation, $\mathrm{Cut_{CS}}$ follows the rule R, and schematically looks as follows:

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \Gamma \Rightarrow \Delta', \theta, \varphi\end{array} \qquad \left.\dfrac{\begin{array}{c}\vdots\,\pi_3\\ \varphi, \Gamma, \Pi \Rightarrow \Delta', \Lambda\end{array}}{\varphi, \Gamma \Rightarrow \Delta', \theta}\, R\right\}\pi_2}{\Gamma \Rightarrow \Delta', \theta}\,\mathrm{Cut_{CS}}$$

This only covers the case where R is a right rule with one premise, A is not the principal formula of R, and the formula θ which *is* principal occurs in the succedent of the right premise of $\mathrm{Cut_{CS}}$. If θ occurs in the antecedent (i.e., R is a left rule) or in the left premise of $\mathrm{Cut_{CS}}$, it would look different, of course. The formulas in Π and Λ represent the active formulas of rule R.

We turn this order around and consider a derivation where R follows $\mathrm{Cut}_{\mathrm{CS}}$:

$$\dfrac{\dfrac{\vdots\,\pi_1'}{\Gamma, \Pi \Rightarrow \Delta', \theta, \Lambda, \varphi} \qquad \dfrac{\vdots\,\pi_3'}{\varphi, \Gamma, \Pi \Rightarrow \Delta', \theta, \Lambda}\,\mathrm{Cut}_{\mathrm{CS}}}{\dfrac{\Gamma, \Pi \Rightarrow \Delta', \theta, \Lambda}{\Gamma, \Rightarrow \Delta', \theta, \theta}\,R}$$

We use height-preserving weakening to obtain π_1' and π_3' from π_1 and π_3, respectively.

Because we switch the order of $\mathrm{Cut}_{\mathrm{CS}}$ and R, this is referred to as "permuting a cut upwards across R." Doing so decreases the height of the derivation ending in the right premise of $\mathrm{Cut}_{\mathrm{CS}}$, and so decreases the cut height. That is, the sub-derivations ending in the new $\mathrm{Cut}_{\mathrm{CS}}$ can be assumed to have height no larger than π_1 and π_3, so the cut height is reduced (we've removed one inference on the right). Now the induction hypothesis applies, which tells us that the $\mathrm{Cut}_{\mathrm{CS}}$ inference can be removed. Finally, we remove the extra copy of θ using the admissibility of CR.

Case C: At least one of π_1 and π_2 ends in a one-premise rule R in which φ is not principal. There are a total of 18 cases, depending on whether φ is not principal in the left or the right premise of $\mathrm{Cut}_{\mathrm{CS}}$, and depending on which of the nine one-premise rules ($\neg$L, $\neg$R, $\vee$R, $\wedge$L, $\to$R, and the four quantifier rules) the rule R is. We only consider two examples: where φ is not principal in the last inference of π_2, which ends in $\to$R or in $\exists$L. The other cases are analogous and left as exercises.

Suppose π ends in:

$$\dfrac{\dfrac{\vdots\,\pi_1}{\Gamma \Rightarrow \Delta', \psi \to \chi, \varphi} \qquad \left.\dfrac{\dfrac{\vdots\,\pi_3}{\varphi, \psi, \Gamma \Rightarrow \Delta', \chi}}{\varphi, \Gamma \Rightarrow \Delta', \psi \to \chi}\,{\to}\mathrm{R}\right\}\pi_2}{\Gamma \Rightarrow \Delta', \psi \to \chi}\,\mathrm{Cut}_{\mathrm{CS}}$$

where $\Delta = \Delta', \psi \to \chi$. We apply height-preserving weakening (**??**) to π_1 to obtain a derivation π_1' of $\psi, \Gamma \Rightarrow \Delta', \psi \to \chi, \chi, \varphi$ (we weaken with ψ on the left and χ on the right). Similarly, we apply **??** to π_3 to get a derivation π_3' of $\varphi, \psi, \Gamma \Rightarrow \Delta', \psi \to \chi, \chi$ (by weakening with $\psi \to \chi$ on the right). The induction hypothesis applies to

$$\dfrac{\dfrac{\vdots\,\pi_1'}{\psi, \Gamma \Rightarrow \Delta', \psi \to \chi, \chi, \varphi} \qquad \dfrac{\vdots\,\pi_3'}{\varphi, \psi, \Gamma \Rightarrow \Delta', \psi \to \chi, \chi}\,\mathrm{Cut}_{\mathrm{CS}}}{\psi, \Gamma \Rightarrow \Delta', \psi \to \chi, \chi}$$

with cut formula φ: This derivation has the same cut rank and a lower cut height than π. This gives us a derivation π_4 in **G3c** without $\mathrm{Cut}_{\mathrm{CS}}$ of $\psi, \Gamma \Rightarrow \Delta', \psi \to \chi, \chi$. Apply $\to$R to get a derivation of $\Gamma \Rightarrow \Delta', \psi \to \chi, \psi \to \chi$:

$$\dfrac{\begin{array}{c}\vdots\,\pi_4\\ \psi, \Gamma \Rightarrow \Delta', \psi \to \chi, \chi\end{array}}{\Gamma \Rightarrow \Delta', \psi \to \chi, \psi \to \chi}\ {\to}\mathrm{R}$$

To get a derivation π' of $\Gamma \Rightarrow \Delta$, i.e., $\Gamma \Rightarrow \Delta', \psi \to \chi$, apply **??** (the admissibility of contraction).

Now suppose π ends in:

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \exists x\, \psi(x), \Gamma' \Rightarrow \Delta, \varphi\end{array} \qquad \left.\dfrac{\begin{array}{c}\vdots\,\pi_3\\ \varphi, \psi(c), \Gamma' \Rightarrow \Delta\end{array}}{\varphi, \exists x\, \psi(x), \Gamma' \Rightarrow \Delta}\ \exists\mathrm{L}\right\}\pi_2}{\Gamma' \Rightarrow \Delta}\ \mathrm{Cut}_{\mathrm{CS}}$$

We again apply the inductive hypothesis to

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1[\psi(c) \Rightarrow\,]\\ \exists x\, \psi(x), \psi(c), \Gamma' \Rightarrow \Delta, \varphi\end{array} \qquad \begin{array}{c}\pi_3[\exists x\, \psi(x) \Rightarrow\,]\,\vdots\\ \varphi, \exists x\, \psi(x), \psi(c), \Gamma' \Rightarrow \Delta\end{array}}{\exists x\, \psi(x), \psi(c), \Gamma' \Rightarrow \Delta}\ \mathrm{Cut}_{\mathrm{CS}}}{\exists x\, \psi(x), \exists x\, \psi(x), \Gamma' \Rightarrow \Delta}\ \exists\mathrm{L}$$

Note that the eigenvariable condition is satisfied, since the eigenvariable condition on $\exists$L in π_2 guarantees that c does not occur in $\exists x\, \psi(x)$, Γ', or Δ. Finally, apply **??**.

Case D: At least one of π_1 and π_2 ends in a two-premise rule R in which φ is not principal. There are 6 cases. We consider the case where φ is not principal in the last inference of π_2, which ends in $\wedge$R; the other cases are analogous.

Assume π ends in:

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \Gamma \Rightarrow \Delta', \psi \wedge \chi, \varphi\end{array} \qquad \left.\dfrac{\begin{array}{c}\vdots\,\pi_3\\ \varphi, \Gamma \Rightarrow \Delta', \psi\end{array} \qquad \begin{array}{c}\vdots\,\pi_4\\ \varphi, \Gamma \Rightarrow \Delta', \chi\end{array}}{\varphi, \Gamma \Rightarrow \Delta', \psi \wedge \chi}\ \wedge\mathrm{R}\right\}\pi_2}{\Gamma \Rightarrow \Delta}\ \mathrm{Cut}_{\mathrm{CS}}$$

The inductive hypothesis applies to the derivations

$$\dfrac{\begin{array}{c}\vdots\,\pi_1'\\ \Gamma \Rightarrow \Delta', \psi \wedge \chi, \psi, \varphi\end{array} \qquad \begin{array}{c}\vdots\,\pi_3'\\ \varphi, \Gamma \Rightarrow \Delta', \psi \wedge \chi, \psi\end{array}}{\Gamma \Rightarrow \Delta', \psi \wedge \chi, \psi}\,\mathrm{Cut_{CS}}$$

and

$$\dfrac{\begin{array}{c}\vdots\,\pi_1''\\ \Gamma \Rightarrow \Delta, \psi \wedge \chi, \chi, \varphi\end{array} \qquad \begin{array}{c}\vdots\,\pi_4'\\ \varphi, \Gamma \Rightarrow \Delta', \psi \wedge \chi, \chi\end{array}}{\Gamma \Rightarrow \Delta', \psi \wedge \chi, \chi}\,\mathrm{Cut_{CS}}$$

Here, the **G3c** derivations π_1' and π_1'' are obtained by height-preserving weakening (**??**) from π_1 (once weakening on the right with ψ, and once with χ). The derivations π_3' and π_4' are obtained from π_3 and π_4 also by height-preserving weakening from π_3 and π_4, respectively (by weakening with $\psi \wedge \chi$ on the right).

The inductive hypothesis gives us derivations π_5 and π_6 in **G3c** of $\Gamma \Rightarrow \Delta', \psi \wedge \chi, \psi$ and $\Gamma \Rightarrow \Delta', \psi \wedge \chi, \chi$. We combine these into a derivation of $\Gamma \Rightarrow \Delta', \psi \wedge \chi, \psi \wedge \chi$ using the rule $\wedge$R:

$$\dfrac{\begin{array}{c}\vdots\,\pi_5\\ \Gamma \Rightarrow \Delta', \psi \wedge \chi, \psi\end{array} \qquad \begin{array}{c}\vdots\,\pi_6\\ \Gamma \Rightarrow \Delta', \psi \wedge \chi, \chi\end{array}}{\Gamma \Rightarrow \Delta, \psi \wedge \chi, \psi \wedge \chi}\,\wedge\mathrm{R}$$

To get a derivation of $\Gamma \Rightarrow \Delta$, i.e., $\Gamma \Rightarrow \Delta', \psi \wedge \chi$, apply **??** (the admissibility of contraction).

Case E: Reducing cuts. The final case concerns the situation where both π_1 and π_2 end in inferences in which φ is principal. There are six cases, one for each possible main operator of φ.

In each case, we turn a $\mathrm{Cut_{CS}}$ with cut formula φ into one or more cuts on sub-formulas of φ, namely on the active formulas of the inferences of which φ is the principal formula. These cases are therefore often referred to as "reduction" or "conversion" of the cut inference.

1. If $\varphi \equiv \neg\psi$ then π ends in

$$\pi_1\left\{\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1'\\ \psi, \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta, \neg\psi}\,\neg\mathrm{R} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_2'\\ \Gamma \Rightarrow \Delta, \psi\end{array}}{\neg\psi, \Gamma \Rightarrow \Delta}\,\neg\mathrm{L}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut_{CS}}\right\}\pi_2$$

By inductive hypothesis, $\mathrm{Cut}_{\mathrm{CS}}$ can be eliminated from

$$\dfrac{\begin{array}{cc}\vdots\,\pi_2' & \vdots\,\pi_1' \\ \Gamma \Rightarrow \Delta, \psi & \psi, \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

2. If $\varphi \equiv (\psi \vee \chi)$ then π ends in

$$\pi_1\left\{\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1' \\ \Gamma \Rightarrow \Delta, \psi, \chi\end{array}}{\Gamma \Rightarrow \Delta, \psi \vee \chi}\,\vee\mathrm{R} \qquad \dfrac{\begin{array}{cc}\vdots\,\pi_2' & \vdots\,\pi_2'' \\ \psi, \Gamma \Rightarrow \Delta & \chi, \Gamma \Rightarrow \Delta\end{array}}{\psi \vee \chi, \Gamma \Rightarrow \Delta}\,\vee\mathrm{L}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}\right\}\pi_2$$

Consider the derivation ending in $\mathrm{Cut}_{\mathrm{CS}}$,

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1' & \vdots\,\pi_2''' \\ \Gamma \Rightarrow \Delta, \psi, \chi & \chi, \Gamma \Rightarrow \Delta, \psi\end{array}}{\Gamma \Rightarrow \Delta, \psi}\,\mathrm{Cut}_{\mathrm{CS}}$$

where π_2''' is obtained from π_2'' by height-preserving weakening. It has cut rank $< \mathrm{cr}(\pi)$ as $\mathrm{dp}(\chi) < \mathrm{dp}(\psi \vee \chi)$, so the inductive hypothesis yields a derivation in **G3c** of π_1'' of $\Gamma \Rightarrow \Delta, \psi$. The derivation ending in $\mathrm{Cut}_{\mathrm{CS}}$,

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1'' & \vdots\,\pi_2' \\ \Gamma \Rightarrow \Delta, \psi & \psi, \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

has cut rank $= \mathrm{dp}(\psi) < \mathrm{dp}(\psi \vee \chi)$, so the inductive hypothesis applies again, and yields a derivation π' of $\Gamma \Rightarrow \Delta$.

3. $\varphi \equiv (\psi \wedge \chi)$. Exercise.

4. If $\varphi \equiv (\psi \to \chi)$ then π ends in

$$\pi_1\left\{\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1' \\ \psi, \Gamma \Rightarrow \Delta, \chi\end{array}}{\Gamma \Rightarrow \Delta, \psi \to \chi}\,\to\mathrm{R} \qquad \dfrac{\begin{array}{cc}\vdots\,\pi_2' & \vdots\,\pi_2'' \\ \Gamma \Rightarrow \Delta, \psi & \chi, \Gamma \Rightarrow \Delta\end{array}}{\psi \to \chi, \Gamma \Rightarrow \Delta}\,\to\mathrm{L}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}\right\}\pi_2$$

The derivation ending in $\mathrm{Cut}_{\mathrm{CS}}$,

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1' & \vdots\,\pi_2'' \\ \psi, \Gamma \Rightarrow \Delta, \chi & \chi, \Gamma \Rightarrow \Delta\end{array}}{\psi, \Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

has lower cut rank than π. The inductive hypthesis yields a derivation π_1'' of $\psi, \Gamma \Rightarrow \Delta$. Now consider the derivation ending in $\mathrm{Cut}_{\mathrm{CS}}$,

$$\dfrac{\begin{array}{cc}\vdots\,\pi_2' & \vdots\,\pi_1'' \\ \Gamma \Rightarrow \Delta, \psi & \psi, \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

Its cut rank is also less than the cut rank of π, so the induction hypothesis yields a **G3c**-derivation π' of $\Gamma \Rightarrow \Delta$.

5. If $\varphi \equiv \forall x\, \psi(x)$ then π ends in

$$\dfrac{\left\{\dfrac{\dfrac{\vdots\,\pi_1'}{\Gamma \Rightarrow \Delta, \psi(c)}}{\Gamma \Rightarrow \Delta, \forall x\, \psi(x)}\,\forall\mathrm{R} \quad \dfrac{\dfrac{\vdots\,\pi_2'}{\psi(t), \forall x\, \psi(x), \Gamma \Rightarrow \Delta}}{\forall x\, \psi(x), \Gamma \Rightarrow \Delta}\,\forall\mathrm{L}\right\}\pi_2}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

By inductive hypothesis, $\mathrm{Cut}_{\mathrm{CS}}$ can be eliminated from

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1'' & \vdots\,\pi_2' \\ \psi(t), \Gamma \Rightarrow \Delta, \forall x\, \psi(x) & \psi(t), \forall x\, \psi(x), \Gamma \Rightarrow \Delta\end{array}}{\psi(t), \Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

where π_1'' is obtained from π_1 (not π_1'!) by height-preserving weakening. (The derivation has the same cut rank but lower cut height.) This gives us a derivation π_2'' of $\psi(t), \Gamma \Rightarrow \Delta$.

The derivation π_1' has end-sequent $\Gamma \Rightarrow \Delta, \psi(c)$, where c is the eigenvariable of a $\forall$R inference. So c does not occur in $\psi(x)$, Γ, or Δ. We assume the derivation π is regular, so c is the eigenvariable of only the following $\forall$R inference and only occurs above it, i.e., only occurs in π_1' and is not an eigenvariable of an inference in π_1'. Since c does not occur in π_2, it cannot occur in t. By **??**, the derivation $\pi_1'[t/c]$ is a derivation of $\Gamma \Rightarrow \Delta, \psi(t)$. We now apply the inductive hypothesis to the derivation

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1'[t/c] & \vdots\,\pi_2'' \\ \Gamma \Rightarrow \Delta, \psi(t) & \psi(t), \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

(The induction hypothesis applies since the cut formula is $\psi(t)$, so the derivation has lower cut rank than π.)

6. We leave the case $\varphi \equiv \exists x\, \psi(x)$ as an exercise. □

Problem cut.1. Verify the sub-case of case D in the proof of Lemma cut.2 where φ is not principal in the last inference of π_1, which ends in $\to$L. (Note: part of this exercise is to clearly state what you have to prove, i.e., what form π has in this case.)

Problem cut.2. Verify the sub-case of case D in the proof of Lemma cut.2 where φ is not principal in the last inference of π_1, which ends in $\forall$L.

Problem cut.3. Verify the sub-case of case E in the proof of Lemma cut.2 where φ is principal in the last inferences of both π_1 and π_2 and $\varphi \equiv (\psi \wedge \chi)$.

Problem cut.4. Verify the sub-case of case E in the proof of Lemma cut.2 where φ is principal in the last inferences of both π_1 and π_2 and $\varphi \equiv \exists x\, \psi(x)$.

Note that in case E the cut height of the derivation ending in Cut_{CS} to which we apply the inductive hypothesis can be larger than the cut height of the original proof. E.g., in the case where $\varphi \equiv (\psi \to \chi)$ we turned π into a derivation involving two Cut_{CS} inferences,

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \psi \quad (\pi_2')}{\psi, \Gamma \Rightarrow \Delta}\,\text{Cut}_{\text{CS}} \qquad \dfrac{\psi, \Gamma \Rightarrow \Delta, \chi \quad (\pi_1') \qquad \chi, \Gamma \Rightarrow \Delta \quad (\pi_2'')}{\psi, \Gamma \Rightarrow \Delta}\,\text{Cut}_{\text{CS}}\ (\pi_1'')}{\psi, \Gamma \Rightarrow \Delta}\,\text{Cut}_{\text{CS}}$$

The top Cut_{CS} between π_1' and π_2'' has both lower cut rank and lower cut height than π. But the derivation π_1''—the result of applying the inductive hypothesis—no longer has lower cut height than π. Since the cut formula of the lower Cut_{CS} is ψ, its cut *rank*, however, is lower than the cut rank of π.

Lemma cut.2 establishes that we can eliminate a single topmost Cut_{CS} inference from a derivation. We can show that we can eliminate any number of Cut_{CS} inferences from a derivation by successively applying it to the topmost sub-derivations ending in Cut_{CS}.

Theorem cut.3. *Any derivation π in* **G3c** $+ \text{Cut}_{\text{CS}}$ *can be transformed into a derivation in* **G3c**. *pt:cut:top: thm:cut-elim-G3c-topmost*

Proof. By induction on the number n of Cut_{CS} inferences in π. If $n = 0$, there is nothing to do: π is already a derivation in **G3c**. Otherwise, consider the sub-derivation π' ending in a top-most Cut_{CS} inference. It contains a single Cut_{CS} rule as the last inference. Apply Lemma cut.2 to transform it into a Cut_{CS}-free proof π'', and replace the sub-derivation π' by π''. The result is a derivation containing $n - 1$ Cut_{CS} inferences, and the induction hypothesis applies. □

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Bibliography