cut.1 Removing Largest Cuts

pt:cut:inv:sec

The proof of **??** shown that if **G3c** derives the conclusion of a logical rule (other than ∃R and ∀L), it also derives each of the premises of the rule without increasing the height of the proof. We can do better and show that this holds also for $\mathbf{G3c} + \mathrm{Cut}_{\mathrm{CS}}$.

As before, we take the *cut rank* of a $\mathrm{Cut}_{\mathrm{CS}}$ inference to be the depth of its cut formula.

pt:cut:inv:lem:inv-G3c-cut

Lemma cut.1. *If* $\mathbf{G3c} + \mathrm{Cut}_{\mathrm{CS}}$ *derives the conclusion of a logical rule* R *other than* ∃R *or* ∀L *using a derivation* π *then it also derives each premise with derivation* π' *(or derivations* π', π'' *depending on whether the rule has one or two premises). Moreover, (a)* $\mathrm{ht}(\pi') \leq \mathrm{ht}(\pi)$ *(and* $\mathrm{ht}(\pi'') \leq \mathrm{ht}(\pi)$*) and (b) the number of* $\mathrm{Cut}_{\mathrm{CS}}$ *inferences as well as their ranks is the same in* π' *(and* π''*) as in* π*.*

Proof. By inspection of the proofs of **????**. That proof was by induction on $\mathrm{ht}(\pi)$. In the base case, we replaced an axiom by another axiom. So conditions (a) and (b) are trivially satisfied. If the last rule of π was R, its immediate sub-derivations are the derivations π_i we want, and (a) and (b) are both satisfied. In all other cases, we applied the inductive hypothesis to the immediate sub-derivations of π, i.e., those leading to the premise(s) of the last inference R, and then applied that same rule R to the results. The conditions (a) and (b) hold for the immediate sub-derivations of the new proof, and so (a) and (b) hold for the entire proof. It remains to verify the case where the last inference is $\mathrm{Cut}_{\mathrm{CS}}$. We do this again for one example only: suppose π is a derivation of the conclusion $\psi \wedge \chi, \Gamma \Rightarrow \Delta$ of ∧L, and ends in $\mathrm{Cut}_{\mathrm{CS}}$:

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1 & \vdots\,\pi_2\\ \psi\wedge\chi,\Gamma \Rightarrow \Delta,\varphi & \varphi,\psi\wedge\chi,\Gamma\Rightarrow\Delta\end{array}}{\psi\wedge\chi,\Gamma\Rightarrow\Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

The induction hypothesis applies to π_1 and π_2 (which are also derivations of instances of the conclusion of ∧L) and yields derivations π_1' and π_2' of $\psi, \chi, \Gamma \Rightarrow \Delta, \varphi$ and $\varphi, \psi, \chi, \Gamma \Rightarrow \Delta$, respectively. We combine these using $\mathrm{Cut}_{\mathrm{CS}}$ to obtain π':

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1' & \vdots\,\pi_2'\\ \psi,\chi,\Gamma \Rightarrow \Delta,\varphi & \varphi,\psi,\chi,\Gamma\Rightarrow\Delta\end{array}}{\psi,\chi,\Gamma\Rightarrow\Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

Clearly, (a) and (b) are satisfied. □

Note that this would not work if we used Cut instead of $\mathrm{Cut}_{\mathrm{CS}}$, as then the conclusion of the last derivation would have two copies of ψ, χ, Γ in the antecedent and two copies of Δ in the succedent. We would have to appeal

to the admissibility of contraction, but the proof of **??** uses the invertibility lemma. We'd thus be arguing in a circle.

We can use Lemma cut.1 to give an alternative proof of cut elimination. In this proof we do not successively remove topmost occurrences of $\mathrm{Cut}_{\mathrm{CS}}$ inferences, but occurrences of $\mathrm{Cut}_{\mathrm{CS}}$ inferences of maximal rank. That is, we start with a derivation π in $\mathbf{G3c} + \mathrm{Cut}_{\mathrm{CS}}$ and remove a cut anywhere in it (possibly replacing it with cuts of smaller rank)—and then keep doing that until no cuts remain. The only condition is that above the cut we deal with, no cuts of the same or higher rank occur.

pt:cut:inv: lem:max-cut-red-G3c

Lemma cut.2. *If π is a derivation in $\mathbf{G3c} + \mathrm{Cut}_{\mathrm{CS}}$ ending in a $\mathrm{Cut}_{\mathrm{CS}}$ inference of rank n and otherwise using only rules of $\mathbf{G3c}$ and $\mathrm{Cut}_{\mathrm{CS}}$ inferences of rank $< n$, then there is a proof π' of the same end-sequent in $\mathbf{G3c} + \mathrm{Cut}_{\mathrm{CS}}$ where every $\mathrm{Cut}_{\mathrm{CS}}$ inference has rank $< n$.*

Proof. The derivation π ends in:

$$\dfrac{\begin{array}{cc}\vdots\,\pi_1 & \vdots\,\pi_2\\ \Gamma \Rightarrow \Delta, \varphi & \varphi, \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}_{\mathrm{CS}}$$

We distinguish cases according to the form of φ.

Suppose φ is atomic. Observe that by the condition that all $\mathrm{Cut}_{\mathrm{CS}}$ inferences in π_1 and π_2 must have rank $< \mathrm{dp}(\varphi) = 0$, there can be no $\mathrm{Cut}_{\mathrm{CS}}$ inferences in π_1 or π_2. We show that $\Gamma \Rightarrow \Delta$ has a cut-free proof by induction on $\mathrm{ht}(\pi_2)$. If $\mathrm{ht}(\pi_2) = 0$, then $\varphi, \Gamma \Rightarrow \Delta$ is an axiom. If φ is not principal, some atomic χ is an element of both Γ and Δ, and $\Gamma \Rightarrow \Delta$ itself is an axiom. If φ is principal, then $\Delta = \Delta, \varphi$. In this case, π_1 ends in $\Gamma \Rightarrow \Delta', \varphi, \varphi$. We obtain a cut-free proof of $\Gamma \Rightarrow \Delta', \varphi$ by applying **??** (admissibility of contraction). If $\mathrm{ht}(\pi_2) > 0$ it ends in a rule R. Take a premise of that rule: it has the form $\varphi, \Gamma', \Pi \Rightarrow \Delta', \Lambda$, where Π and Λ are the active formulas of R. **??** applied to π_1 and π_2 yields derivations of $\Gamma, \Gamma', \Pi \Rightarrow \Delta, \Delta', \Lambda, \varphi$ and $\varphi, \Gamma, \Gamma', \Pi \Rightarrow \Delta, \Delta', \Lambda$, to which the induction hypothesis applies. This gives us a derivation of $\Gamma, \Gamma', \Pi \Rightarrow \Delta, \Delta', \Lambda$, for each premise of R. Applying R yields $\Gamma, \Gamma \Rightarrow \Delta, \Delta$, and **??** provides the cut-free proof of $\Gamma \Rightarrow \Delta$.

If φ is not atomic, we distinguish cases according to the main operator of φ. In each case, we apply Lemma cut.1 to obtain derivations of the premises of the left- and right-rules with φ as principal formula. We then proceed as in case E of the proof of **??**.

1. $\varphi \equiv \neg\psi$. The derivations π_1 and π_2 end in $\Gamma \Rightarrow \Delta, \neg\psi$ and $\neg\psi, \Gamma \Rightarrow \Delta$. By Lemma cut.1, there are derivations π_1' and π_2' of $\psi, \Gamma \Rightarrow \Delta$ and

$\Gamma \Rightarrow \Delta, \psi$. The derivation π' is:

$$\dfrac{\begin{array}{cc}\vdots\,\pi'_2 & \vdots\,\pi'_1\\ \Gamma \Rightarrow \Delta, \psi & \psi, \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut_{CS}}$$

2. $\varphi \equiv (\psi \vee \chi)$. The derivations π_1 and π_2 end in $\Gamma \Rightarrow \Delta, \psi \vee \chi$ and $\psi \vee \chi, \Gamma \Rightarrow \Delta$. By Lemma cut.1 (inversion for $\vee$R applied to π_1), there is a derivation π'_1 of $\Gamma \Rightarrow \Delta, \psi, \chi$. By Lemma cut.1 (inversion for $\vee$L applied to π_2), there is a derivation π'_2 of $\psi, \Gamma \Rightarrow \Delta$ and a derivation π''_2 of $\chi, \Gamma \Rightarrow \Delta$. By applying **??** to π''_2 we also get a derivation π'''_2 of $\chi, \Gamma \Rightarrow \Delta, \psi$. The derivation π' is:

$$\dfrac{\dfrac{\begin{array}{cc}\vdots\,\pi'_1 & \vdots\,\pi'''_2\\ \Gamma \Rightarrow \Delta, \psi, \chi & \chi, \Gamma \Rightarrow \Delta, \psi\end{array}}{\Gamma \Rightarrow \Delta, \psi}\,\mathrm{Cut_{CS}} \qquad \begin{array}{c}\vdots\,\pi'_2\\ \psi, \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut_{CS}}$$

3. $\varphi \equiv (\psi \wedge \chi)$. Exercise.

4. $\varphi \equiv (\psi \to \chi)$. The derivations π_1 and π_2 end in $\Gamma \Rightarrow \Delta, \psi \to \chi$ and $\psi \to \chi, \Gamma \Rightarrow \Delta$. By Lemma cut.1 (inversion for $\to$R applied to π_1), there is a derivation π'_1 of $\psi, \Gamma \Rightarrow \Delta, \chi$. By Lemma cut.1 (inversion for $\to$L applied to π_2), there is a derivation π'_2 of $\Gamma \Rightarrow \Delta, \psi$ and a derivation π''_2 of $\chi, \Gamma \Rightarrow \Delta$. By applying **??** to π''_2 we also get a derivation π'''_2 of $\chi, \psi, \Gamma \Rightarrow \Delta$. The derivation π' is:

$$\dfrac{\begin{array}{cc}\vdots\,\pi'_2 & \dfrac{\begin{array}{cc}\vdots\,\pi'_1 & \vdots\,\pi'''_2\\ \psi, \Gamma \Rightarrow \Delta, \chi & \chi, \psi, \Gamma \Rightarrow \Delta\end{array}}{\psi, \Gamma \Rightarrow \Delta}\,\mathrm{Cut_{CS}}\\ \Gamma \Rightarrow \Delta, \psi & \end{array}}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut_{CS}}$$

5. $\varphi \equiv \forall x\, \psi(x)$. The derivations π_1 and π_2 end in $\Gamma \Rightarrow \Delta, \forall x\, \psi(x)$ and $\forall x\, \psi(x), \Gamma \Rightarrow \Delta$. By Lemma cut.1, there is a derivation π'_1 of $\Gamma \Rightarrow \Delta, \psi(c)$.

   Now consider the derivation π_2. It ends in $\forall x\, \psi(x), \Gamma \Rightarrow \Delta$. Moving upward from the end-sequent of π_2, mark every occurrence of $\forall x\, \psi(x)$ on the left of a sequent that "leads to" the occurrence of $\forall x\, \psi(x)$ in the end-sequent of π_2, that is: (a) Mark $\forall x\, \psi(x)$ in the end-sequent of π_2. (b) If $\forall x\, \psi(x)$ occurs in the context of the conclusion of a rule and is marked, mark a corresponding occurrence in the premise(s) of the rule. (c) If $\forall x\, \psi(x)$ is principal in the conclusion of a $\forall$L rule and is marked,

mark the corresponding active occurrences of $\forall x\,\psi(x)$ and $\psi(t)$ in the premise.

Now consider all these marked occurrences of $\forall x\,\psi(x)$. None of them are active in a rule other than $\forall$L (only active formulas of $\forall$L get marked). Delete all marked occurrences of $\forall x\,\psi(x)$ in π_2. If this is a correct derivation, the marked occurrences of $\forall x\,\psi(x)$ were never active; they all came directly from axioms. The derivation ends in $\Gamma \Rightarrow \Delta$ and we can replace π by it. We have a removed a cut of maximal rank.

Otherwise, what we have is not a correct proof because we have deleted formulas $\forall x\,\psi(x)$ which were active in some $\forall$L inferences. We have turned these into "inferences" of the form

$$\dfrac{\begin{array}{c}\vdots\\ \pi_t\end{array}}{\dfrac{\psi(t), \Pi \Rightarrow \Lambda}{\Pi \Rightarrow \Lambda}\,\forall\mathrm{L}}$$

Replace each topmost such "inference" by

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1'[t/c][\Pi \Rightarrow \Lambda]\end{array}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda, \psi(t)} \qquad \dfrac{\begin{array}{c}\vdots\\ \pi_t[\Gamma \Rightarrow \Delta]\end{array}}{\psi(t), \Gamma, \Pi \Rightarrow \Delta, \Lambda}\,\mathrm{Cut_{CS}}$$
$$\Gamma, \Pi \Rightarrow \Delta, \Lambda$$

and add Γ to the left and Δ to the right of every sequent below it. Repeat until all $\forall$L "inferences" with a marked $\psi(t)$ in the premise have been replaced by a $\mathrm{Cut_{CS}}$ inference on some $\psi(t)$. The end-sequent of the resulting derivation (and it now is a correct derivation) is of the form $\Gamma, \ldots, \Gamma \Rightarrow \Delta, \ldots, \Delta$. Apply admissibility of contraction to turn it into a derivation of $\Gamma \Rightarrow \Delta$ and replace π with it. We have replaced one cut on $\forall x\,\psi(x)$ with (possibly many) cuts on $\psi(t)$, but these are all of lower rank. Any cuts already occurring in π_1 or π_2 remain present, but they are also all of lower rank. □

Problem cut.1. Verify the case where $\varphi \equiv (\psi \wedge \chi)$ in the proof of Lemma cut.1. That is, show that if there is a derivation π ending in

$$\dfrac{\begin{array}{c}\vdots\\ \pi_1\end{array}}{\Gamma \Rightarrow \Delta, \psi \wedge \chi} \qquad \dfrac{\begin{array}{c}\vdots\\ \pi_2\end{array}}{\psi \wedge \chi, \Gamma \Rightarrow \Delta}\,\mathrm{Cut_{CS}}$$
$$\Gamma \Rightarrow \Delta$$

with π_1 and π_2 containing only cuts of rank $< \mathrm{dp}(\psi \wedge \chi)$ then there is a derivation π' of $\Gamma \Rightarrow \Delta$ containing only cuts of rank $< \mathrm{dp}(\psi \wedge \chi)$.

?? establishes that we can replace a $\mathrm{Cut}_{\mathrm{CS}}$ inference of highest rank in a derivation by $\mathrm{Cut}_{\mathrm{CS}}$ inferences of lower rank. We can again use this lemma to show that we can eliminate any number of $\mathrm{Cut}_{\mathrm{CS}}$ inferences from a derivation by successively applying it to the topmost sub-derivations ending in maximal $\mathrm{Cut}_{\mathrm{CS}}$.

pt:cut:inv: thm:cut-elim-G3c-largest

Theorem cut.3. *Any derivation π in* **G3c** $+ \mathrm{Cut}_{\mathrm{CS}}$ *can be transformed into a proof in* **G3c**.

Proof. First we prove that all cuts of maximal rank in π can be removed. Let d be the maximal rank of cuts occuring in π, and let n be the number of cuts of rank d. The proof is by induction on n. If $n = 0$ there is nothing to prove. If $n > 0$, pick a topmost cut of rank d, and let π_1 be the sub-derivation ending in it. By Lemma cut.2, there is a derivation π_1' of the same end-sequent with all cuts of rank $< d$. Replace π_1 by π_1' in π. This results in a derivation containing $n - 1$ cuts of rank d, and the induction hypothesis applies.

Now the result follows by induction on d. If $d = 0$, all cuts are atomic, and by the previous result can all be removed. If $d > 0$, the previous result yields a proof in which all cuts of rank d have been replaced by cuts of rank $< d$. Since d was the maximum rank of cuts in π, we have proof in which the maximal rank of cuts is $< d$, and the induction hypothesis applies. □

Photo Credits

Bibliography