tab.1 Soundness for Additional Rules

nml:tab:msn:sec We say a rule is sound for a class of models if, whenever a branch in a tableau is satisfiable in a model from that class, the branch resulting from applying the rule is also satisfiable in a model from that class.

nml:tab:msn:prop:soundness-T **Proposition tab.1.** T$\Box$ *and* T$\Diamond$ *are sound for reflexive models.*

Proof.

1. The branch is expanded by applying T$\Box$ to $\sigma\,\mathbb{T}\,\Box\psi \in \Gamma$: This results in a new signed formula $\sigma\,\mathbb{T}\,\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$. Since R is reflexive, we know that $Rf(\sigma)f(\sigma)$. Hence, $\mathfrak{M}, f(\sigma) \Vdash \psi$, i.e., $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{T}\,\psi$.

2. The branch is expanded by applying T$\Diamond$ to $\sigma\,\mathbb{F}\,\Diamond\psi \in \Gamma$: This results in a new signed formula $\sigma\,\mathbb{F}\,\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \nVdash \Diamond\psi$. Since R is reflexive, we know that $Rf(\sigma)f(\sigma)$. Hence, $\mathfrak{M}, f(\sigma) \nVdash \psi$, i.e., $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{F}\,\psi$. □

nml:tab:msn:prop:soundness-D **Proposition tab.2.** D$\Box$ *and* D$\Diamond$ *are sound for serial models.*

Proof.

1. The branch is expanded by applying D$\Box$ to $\sigma\,\mathbb{T}\,\Box\psi \in \Gamma$: This results in a new signed formula $\sigma\,\mathbb{T}\,\Diamond\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$. Since R is serial, there is a $w \in W$ such that $Rf(\sigma)w$. Then $\mathfrak{M}, w \Vdash \psi$, and hence $\mathfrak{M}, f(\sigma) \Vdash \Diamond\psi$. So, $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{T}\,\Diamond\psi$.

2. The branch is expanded by applying D$\Diamond$ to $\sigma\,\mathbb{F}\,\Diamond\psi \in \Gamma$: This results in a new signed formula $\sigma\,\mathbb{F}\,\Box\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \nVdash \Diamond\psi$. Since R is serial, there is a $w \in W$ such that $Rf(\sigma)w$. Then $\mathfrak{M}, w \nVdash \psi$, and hence $\mathfrak{M}, f(\sigma) \nVdash \Box\psi$. So, $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{F}\,\Box\psi$. □

nml:tab:msn:prop:soundness-B **Proposition tab.3.** B$\Box$ *and* B$\Diamond$ *are sound for symmetric models.*

Proof.

1. The branch is expanded by applying B$\Box$ to $\sigma.n\,\mathbb{T}\,\Box\psi \in \Gamma$: This results in a new signed formula $\sigma\,\mathbb{T}\,\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$, we know that $Rf(\sigma)f(\sigma.n)$. Since R is symmetric, $Rf(\sigma.n)f(\sigma)$. Since $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$, $\mathfrak{M}, f(\sigma) \Vdash \psi$. Hence, $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{T}\,\psi$.

2. The branch is expanded by applying B$\Diamond$ to $\sigma.n\,\mathbb{F}\,\Diamond\psi \in \Gamma$: This results in a new signed formula $\sigma\,\mathbb{F}\,\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma.n) \nVdash \Diamond\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$, we know that $Rf(\sigma)f(\sigma.n)$. Since R is symmetric, $Rf(\sigma.n)f(\sigma)$. Since $\mathfrak{M}, f(\sigma.n) \nVdash \Diamond\psi$, $\mathfrak{M}, f(\sigma) \nVdash \psi$. Hence, $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{F}\,\psi$. □

Proposition tab.4. $4\Box$ *and* $4\Diamond$ *are sound for transitive models.* nml:tab:msn: prop:soundness-4

Proof.
1. The branch is expanded by applying $4\Box$ to $\sigma\,\mathbb{T}\,\Box\psi \in \Gamma$: This results in a new signed formula $\sigma.n\,\mathbb{T}\,\Box\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$ and $\sigma.n$ must be used, we know that $Rf(\sigma)f(\sigma.n)$. Now let w be any world such that $Rf(\sigma.n)w$. Since R is transitive, $Rf(\sigma)w$. Since $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$, $\mathfrak{M}, w \Vdash \psi$. Hence, $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$, and $\mathfrak{M}, f$ satisfies $\sigma.n\,\mathbb{T}\,\Box\psi$.
2. The branch is expanded by applying $4\Diamond$ to $\sigma\,\mathbb{F}\,\Diamond\psi \in \Gamma$: This results in a new signed formula $\sigma.n\,\mathbb{F}\,\Diamond\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \nVdash \Diamond\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$ and $\sigma.n$ must be used, we know that $Rf(\sigma)f(\sigma.n)$. Now let w be any world such that $Rf(\sigma.n)w$. Since R is transitive, $Rf(\sigma)w$. Since $\mathfrak{M}, f(\sigma) \nVdash \Diamond\psi$, $\mathfrak{M}, w \nVdash \psi$. Hence, $\mathfrak{M}, f(\sigma.n) \nVdash \Diamond\psi$, and $\mathfrak{M}, f$ satisfies $\sigma.n\,\mathbb{F}\,\Diamond\psi$. □

Proposition tab.5. $4\mathrm{r}\Box$ *and* $4\mathrm{r}\Diamond$ *are sound for euclidean models.* nml:tab:msn: prop:soundness-4r

Proof.
1. The branch is expanded by applying $4\mathrm{r}\Box$ to $\sigma.n\,\mathbb{T}\,\Box\psi \in \Gamma$: This results in a new signed formula $\sigma\,\mathbb{T}\,\Box\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$, we know that $Rf(\sigma)f(\sigma.n)$. Now let w be any world such that $Rf(\sigma)w$. Since R is euclidean, $Rf(\sigma.n)w$. Since $\mathfrak{M}, f(\sigma).n \Vdash \Box\psi$, $\mathfrak{M}, w \Vdash \psi$. Hence, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$, and $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{T}\,\Box\psi$.
2. The branch is expanded by applying $4\mathrm{r}\Diamond$ to $\sigma.n\,\mathbb{F}\,\Diamond\psi \in \Gamma$: This results in a new signed formula $\sigma\,\mathbb{T}\,\Box\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma.n) \nVdash \Diamond\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$, we know that $Rf(\sigma)f(\sigma.n)$. Now let w be any world such that $Rf(\sigma)w$. Since R is euclidean, $Rf(\sigma.n)w$. Since $\mathfrak{M}, f(\sigma).n \nVdash \Diamond\psi$, $\mathfrak{M}, w \nVdash \psi$. Hence, $\mathfrak{M}, f(\sigma) \nVdash \Diamond\psi$, and $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{F}\,\Diamond\psi$. □

Corollary tab.6. *The tableau systems given in* **??** *are sound for the respective classes of models.* nml:tab:msn: cor:soundness-logics

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Bibliography