

## tab.1 Soundness for Additional Rules

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sec

We say a rule is sound for a class of models if, whenever a branch in a **tableau** is satisfiable in a model from that class, the branch resulting from applying the rule is also satisfiable in a model from that class.

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prop:soundness-T

**Proposition tab.1.**  $T\Box$  and  $T\Diamond$  are sound for reflexive models.

*Proof.* 1. The branch is expanded by applying  $T\Box$  to  $\sigma T\Box\psi \in \Gamma$ : This results in a new **signed formula**  $\sigma T\psi$  on the branch. Suppose  $\mathfrak{M}, f \Vdash \Gamma$ , in particular,  $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$ . Since  $R$  is reflexive, we know that  $Rf(\sigma)f(\sigma)$ . Hence,  $\mathfrak{M}, f(\sigma) \Vdash \psi$ , i.e.,  $\mathfrak{M}, f$  satisfies  $\sigma T\psi$ .

2. The branch is expanded by applying  $T\Diamond$  to  $\sigma T\Diamond\psi \in \Gamma$ : This results in a new **signed formula**  $\sigma F\psi$  on the branch. Suppose  $\mathfrak{M}, f \Vdash \Gamma$ , in particular,  $\mathfrak{M}, f(\sigma) \not\Vdash \Diamond\psi$ . Since  $R$  is reflexive, we know that  $Rf(\sigma)f(\sigma)$ . Hence,  $\mathfrak{M}, f(\sigma) \not\Vdash \psi$ , i.e.,  $\mathfrak{M}, f$  satisfies  $\sigma F\psi$ .  $\square$

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prop:soundness-D

**Proposition tab.2.**  $D\Box$  and  $D\Diamond$  are sound for serial models.

*Proof.* 1. The branch is expanded by applying  $D\Box$  to  $\sigma T\Box\psi \in \Gamma$ : This results in a new **signed formula**  $\sigma T\Diamond\psi$  on the branch. Suppose  $\mathfrak{M}, f \Vdash \Gamma$ , in particular,  $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$ . Since  $R$  is serial, there is a  $w \in W$  such that  $Rf(\sigma)w$ . Then  $\mathfrak{M}, w \Vdash \psi$ , and hence  $\mathfrak{M}, f(\sigma) \Vdash \Diamond\psi$ . So,  $\mathfrak{M}, f$  satisfies  $\sigma T\Diamond\psi$ .

2. The branch is expanded by applying  $D\Diamond$  to  $\sigma F\Diamond\psi \in \Gamma$ : This results in a new **signed formula**  $\sigma F\Box\psi$  on the branch. Suppose  $\mathfrak{M}, f \Vdash \Gamma$ , in particular,  $\mathfrak{M}, f(\sigma) \not\Vdash \Diamond\psi$ . Since  $R$  is serial, there is a  $w \in W$  such that  $Rf(\sigma)w$ . Then  $\mathfrak{M}, w \not\Vdash \psi$ , and hence  $\mathfrak{M}, f(\sigma) \not\Vdash \Box\psi$ . So,  $\mathfrak{M}, f$  satisfies  $\sigma F\Box\psi$ .  $\square$

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prop:soundness-B

**Proposition tab.3.**  $B\Box$  and  $B\Diamond$  are sound for symmetric models.

*Proof.* 1. The branch is expanded by applying  $B\Box$  to  $\sigma.nT\Box\psi \in \Gamma$ : This results in a new **signed formula**  $\sigma T\psi$  on the branch. Suppose  $\mathfrak{M}, f \Vdash \Gamma$ , in particular,  $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$ . Since  $f$  is an interpretation of prefixes on the branch into  $\mathfrak{M}$ , we know that  $Rf(\sigma)f(\sigma.n)$ . Since  $R$  is symmetric,  $Rf(\sigma.n)f(\sigma)$ . Since  $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$ ,  $\mathfrak{M}, f(\sigma) \Vdash \psi$ . Hence,  $\mathfrak{M}, f$  satisfies  $\sigma T\psi$ .

2. The branch is expanded by applying  $B\Diamond$  to  $\sigma.nF\Diamond\psi \in \Gamma$ : This results in a new **signed formula**  $\sigma F\psi$  on the branch. Suppose  $\mathfrak{M}, f \Vdash \Gamma$ , in particular,  $\mathfrak{M}, f(\sigma.n) \not\Vdash \Diamond\psi$ . Since  $f$  is an interpretation of prefixes on the branch into  $\mathfrak{M}$ , we know that  $Rf(\sigma)f(\sigma.n)$ . Since  $R$  is symmetric,  $Rf(\sigma.n)f(\sigma)$ . Since  $\mathfrak{M}, f(\sigma.n) \not\Vdash \Diamond\psi$ ,  $\mathfrak{M}, f(\sigma) \not\Vdash \psi$ . Hence,  $\mathfrak{M}, f$  satisfies  $\sigma F\psi$ .  $\square$

**Proposition tab.4.**  $4\Box$  and  $4\Diamond$  are sound for transitive models.

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prop:soundness-4*

*Proof.* 1. The branch is expanded by applying  $4\Box$  to  $\sigma\mathbb{T}\Box\psi \in \Gamma$ : This results in a new **signed formula**  $\sigma.n\mathbb{T}\Box\psi$  on the branch. Suppose  $\mathfrak{M}, f \Vdash \Gamma$ , in particular,  $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$ . Since  $f$  is an interpretation of prefixes on the branch into  $\mathfrak{M}$  and  $\sigma.n$  must be used, we know that  $Rf(\sigma)f(\sigma.n)$ . Now let  $w$  be any world such that  $Rf(\sigma.n)w$ . Since  $R$  is transitive,  $Rf(\sigma)w$ . Since  $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$ ,  $\mathfrak{M}, w \Vdash \psi$ . Hence,  $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$ , and  $\mathfrak{M}, f$  satisfies  $\sigma.n\mathbb{T}\Box\psi$ .

2. The branch is expanded by applying  $4\Diamond$  to  $\sigma\mathbb{F}\Diamond\psi \in \Gamma$ : This results in a new **signed formula**  $\sigma.n\mathbb{F}\Diamond\psi$  on the branch. Suppose  $\mathfrak{M}, f \Vdash \Gamma$ , in particular,  $\mathfrak{M}, f(\sigma) \not\Vdash \Diamond\psi$ . Since  $f$  is an interpretation of prefixes on the branch into  $\mathfrak{M}$  and  $\sigma.n$  must be used, we know that  $Rf(\sigma)f(\sigma.n)$ . Now let  $w$  be any world such that  $Rf(\sigma.n)w$ . Since  $R$  is transitive,  $Rf(\sigma)w$ . Since  $\mathfrak{M}, f(\sigma) \not\Vdash \Diamond\psi$ ,  $\mathfrak{M}, w \not\Vdash \psi$ . Hence,  $\mathfrak{M}, f(\sigma.n) \not\Vdash \Diamond\psi$ , and  $\mathfrak{M}, f$  satisfies  $\sigma.n\mathbb{F}\Diamond\psi$ .  $\square$

**Proposition tab.5.**  $4r\Box$  and  $4r\Diamond$  are sound for euclidean models.

*nml:tab:msn:  
prop:soundness-4r*

*Proof.* 1. The branch is expanded by applying  $4r\Box$  to  $\sigma.n\mathbb{T}\Box\psi \in \Gamma$ : This results in a new **signed formula**  $\sigma\mathbb{T}\Box\psi$  on the branch. Suppose  $\mathfrak{M}, f \Vdash \Gamma$ , in particular,  $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$ . Since  $f$  is an interpretation of prefixes on the branch into  $\mathfrak{M}$ , we know that  $Rf(\sigma)f(\sigma.n)$ . Now let  $w$  be any world such that  $Rf(\sigma)w$ . Since  $R$  is euclidean,  $Rf(\sigma.n)w$ . Since  $\mathfrak{M}, f(\sigma).n \Vdash \Box\psi$ ,  $\mathfrak{M}, w \Vdash \psi$ . Hence,  $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$ , and  $\mathfrak{M}, f$  satisfies  $\sigma\mathbb{T}\Box\psi$ .

2. The branch is expanded by applying  $4r\Diamond$  to  $\sigma.n\mathbb{F}\Diamond\psi \in \Gamma$ : This results in a new **signed formula**  $\sigma\mathbb{F}\Diamond\psi$  on the branch. Suppose  $\mathfrak{M}, f \Vdash \Gamma$ , in particular,  $\mathfrak{M}, f(\sigma.n) \not\Vdash \Diamond\psi$ . Since  $f$  is an interpretation of prefixes on the branch into  $\mathfrak{M}$ , we know that  $Rf(\sigma)f(\sigma.n)$ . Now let  $w$  be any world such that  $Rf(\sigma)w$ . Since  $R$  is euclidean,  $Rf(\sigma.n)w$ . Since  $\mathfrak{M}, f(\sigma).n \not\Vdash \Diamond\psi$ ,  $\mathfrak{M}, w \not\Vdash \psi$ . Hence,  $\mathfrak{M}, f(\sigma) \not\Vdash \Diamond\psi$ , and  $\mathfrak{M}, f$  satisfies  $\sigma\mathbb{F}\Diamond\psi$ .  $\square$

**Corollary tab.6.** The *tableau* systems given in ?? are sound for the respective classes of models.

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cor:soundness-logics*

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