

syn.1 Tautological Instances

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sec

A modal-free formula is a tautology if it is true under every truth-value assignment. Clearly, every tautology is true at every world in every model. But for formulas involving \Box and \Diamond , the notion of tautology is not defined. Is it the case, e.g., that $\Box p \vee \neg \Box p$ —an instance of the principle of excluded middle—is valid? The notion of a *tautological instance* helps: a formula that is a substitution instance of a (non-modal) tautology. It is not surprising, but still requires proof, that every tautological instance is valid.

explanation

Definition syn.1. A modal formula ψ is a *tautological instance* if and only if there is a modal-free tautology φ with propositional variables p_1, \dots, p_n and formulas $\theta_1, \dots, \theta_n$ such that $\psi \equiv \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$.

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Lemma syn.2. Suppose φ is a modal-free formula whose propositional variables are p_1, \dots, p_n , and let $\theta_1, \dots, \theta_n$ be modal formulas. Then for any assignment \mathbf{v} , any model $\mathfrak{M} = \langle W, R, V \rangle$, and any $w \in W$ such that $\mathbf{v}(p_i) = \mathbb{T}$ if and only if $\mathfrak{M}, w \Vdash \theta_i$ we have that $\mathbf{v} \models \varphi$ if and only if $\mathfrak{M}, w \Vdash \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$.

Proof. By induction on φ .

1. $\varphi \equiv \perp$: Both $\mathbf{v} \not\models \perp$ and $\mathfrak{M}, w \not\Vdash \perp$.
2. $\varphi \equiv \top$: Both $\mathbf{v} \models \top$ and $\mathfrak{M}, w \Vdash \top$.
3. $\varphi \equiv p_i$:

$$\begin{aligned} \mathbf{v} \models p_i &\Leftrightarrow \mathbf{v}(p_i) = \mathbb{T} \\ &\text{by definition of } \mathbf{v} \models p_i \\ &\Leftrightarrow \mathfrak{M}, w \Vdash \theta_i \\ &\text{by assumption} \\ &\Leftrightarrow \mathfrak{M}, w \Vdash p_i[\theta_1/p_1, \dots, \theta_n/p_n] \\ &\text{since } p_i[\theta_1/p_1, \dots, \theta_n/p_n] \equiv \theta_i. \end{aligned}$$

4. $\varphi \equiv \neg\psi$:

$$\begin{aligned} \mathbf{v} \models \neg\psi &\Leftrightarrow \mathbf{v} \not\models \psi \\ &\text{by definition of } \mathbf{v} \models; \\ &\Leftrightarrow \mathfrak{M}, w \not\Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \\ &\text{by induction hypothesis} \\ &\Leftrightarrow \mathfrak{M}, w \Vdash \neg\psi[\theta_1/p_1, \dots, \theta_n/p_n] \\ &\text{by definition of } \mathbf{v} \models. \end{aligned}$$

5. $\varphi \equiv (\psi \wedge \chi)$:

$$\begin{aligned}
\mathbf{v} \models \psi \wedge \chi &\Leftrightarrow \mathbf{v} \models \psi \text{ and } \mathbf{v} \models \chi \\
&\text{by definition of } \mathbf{v} \models \\
&\Leftrightarrow \mathfrak{M}, w \Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ and} \\
&\quad \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by induction hypothesis} \\
&\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \wedge \chi)[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by definition of } \mathfrak{M}, w \Vdash.
\end{aligned}$$

6. $\varphi \equiv (\psi \vee \chi)$:

$$\begin{aligned}
\mathbf{v} \models \psi \vee \chi &\Leftrightarrow \mathbf{v} \models \psi \text{ or } \mathbf{v} \models \chi \\
&\text{by definition of } \mathbf{v} \models; \\
&\Leftrightarrow \mathfrak{M}, w \Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ or} \\
&\quad \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by induction hypothesis} \\
&\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \vee \chi)[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by definition of } \mathfrak{M}, w \Vdash.
\end{aligned}$$

7. $\varphi \equiv (\psi \rightarrow \chi)$:

$$\begin{aligned}
\mathbf{v} \models \psi \rightarrow \chi &\Leftrightarrow \mathbf{v} \not\models \psi \text{ or } \mathbf{v} \models \chi \\
&\text{by definition of } \mathbf{v} \models \\
&\Leftrightarrow \mathfrak{M}, w \not\Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ or} \\
&\quad \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by induction hypothesis} \\
&\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \rightarrow \chi)[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by definition of } \mathfrak{M}, w \Vdash.
\end{aligned}$$

8. $\varphi \equiv (\psi \leftrightarrow \chi)$:

$$\begin{aligned}
\mathbf{v} \models \psi \leftrightarrow \chi &\Leftrightarrow \text{either } \mathbf{v} \models \psi \text{ and } \mathbf{v} \models \chi \\
&\quad \text{or } \mathbf{v} \not\models \psi \text{ and } \mathbf{v} \not\models \chi \\
&\text{by definition of } \mathbf{v} \models \\
&\Leftrightarrow \text{either } \mathfrak{M}, w \Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ and} \\
&\quad \mathfrak{M}, w \Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\quad \text{or } \mathfrak{M}, w \not\Vdash \psi[\theta_1/p_1, \dots, \theta_n/p_n] \text{ and} \\
&\quad \mathfrak{M}, w \not\Vdash \chi[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by induction hypothesis} \\
&\Leftrightarrow \mathfrak{M}, w \Vdash (\psi \leftrightarrow \chi)[\theta_1/p_1, \dots, \theta_n/p_n] \\
&\text{by definition of } \mathfrak{M}, w \Vdash. \quad \square
\end{aligned}$$

nml:syn:tau: **Proposition syn.3.** *All tautological instances are valid.*
prop:valid-taut

Proof. Contrapositively, suppose φ is such that $\mathfrak{M}, w \not\models \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$, for some model \mathfrak{M} and world w . Define an assignment \mathfrak{v} such that $\mathfrak{v}(p_i) = \mathbb{T}$ if and only if $\mathfrak{M}, w \models \theta_i$ (and \mathfrak{v} assigns arbitrary values to $q \notin \{p_1, \dots, p_n\}$). Then by **Lemma syn.2**, $\mathfrak{v} \not\models \varphi$, so φ is not a tautology. \square

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Bibliography