

syn.1 Schemas and Validity

nml:syn:sch:
sec

Definition syn.1. A *schema* is a set of **formulas** comprising all and only the substitution instances of some modal **formula** χ , i.e.,

$$\{\psi : \exists\theta_1, \dots, \exists\theta_n (\psi = \chi[\theta_1/p_1, \dots, \theta_n/p_n])\}.$$

The **formula** χ is called the *characteristic formula* of the schema, and it is unique up to a renaming of the propositional variables. A **formula** φ is an *instance* of a schema if it is a member of the set.

It is convenient to denote a schema by the meta-linguistic expression obtained by substituting ‘ φ ’, ‘ ψ ’, \dots , for the atomic components of χ . So, for instance, the following denote schemas: ‘ φ ’, ‘ $\varphi \rightarrow \Box\varphi$ ’, ‘ $\varphi \rightarrow (\psi \rightarrow \varphi)$ ’. They correspond to the characteristic **formulas** p , $p \rightarrow \Box p$, $p \rightarrow (q \rightarrow p)$. The schema ‘ φ ’ denotes the set of *all formulas*.

Definition syn.2. A schema is *true* in a model if and only if all of its instances are; and a schema is *valid* if and only if it is true in every model.

nml:syn:sch:
prop:Kvalid

Proposition syn.3. *The following schema K is valid*

$$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi). \quad (\text{K})$$

Proof. We need to show that all instances of the schema are true at every world in every model. So let $\mathfrak{M} = \langle W, R, V \rangle$ and $w \in W$ be arbitrary. To show that a conditional is true at a world we assume the antecedent is true to show that the consequent is true as well. In this case, let $\mathfrak{M}, w \Vdash \Box(\varphi \rightarrow \psi)$ and $\mathfrak{M}, w \Vdash \Box\varphi$. We need to show $\mathfrak{M}, w \Vdash \Box\psi$. So let w' be arbitrary such that Rww' . Then by the first assumption $\mathfrak{M}, w' \Vdash \varphi \rightarrow \psi$ and by the second assumption $\mathfrak{M}, w' \Vdash \varphi$. It follows that $\mathfrak{M}, w' \Vdash \psi$. Since w' was arbitrary, $\mathfrak{M}, w \Vdash \Box\psi$. \square

nml:syn:sch:
prop:Dual-valid

Proposition syn.4. *The following schema DUAL is valid*

$$\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi. \quad (\text{DUAL})$$

Proof. Exercise. \square

Problem syn.1. Prove **Proposition syn.4**.

nml:syn:sch:
prop:soundMP

Proposition syn.5. *If φ and $\varphi \rightarrow \psi$ are true at a world in a model then so is ψ . Hence, the valid **formulas** are closed under modus ponens.*

nml:syn:sch:
prop:valid-instances

Proposition syn.6. *A **formula** φ is valid iff all its substitution instances are. In other words, a schema is valid iff its characteristic **formula** is.*

Valid Schemas	Invalid Schemas
$\Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$	$\Box(\varphi \vee \psi) \rightarrow (\Box\varphi \vee \Box\psi)$
$\Diamond(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Diamond\psi)$	$(\Diamond\varphi \wedge \Diamond\psi) \rightarrow \Diamond(\varphi \wedge \psi)$
$\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$	$\varphi \rightarrow \Box\varphi$
$\Box\varphi \rightarrow \Box(\psi \rightarrow \varphi)$	$\Box\Diamond\varphi \rightarrow \psi$
$\neg\Diamond\varphi \rightarrow \Box(\varphi \rightarrow \psi)$	$\Box\Box\varphi \rightarrow \Box\varphi$
$\Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi)$	$\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$

Table 1: Valid and (or?) invalid schemas.

nml:syn:sch:
tab:valid-invalidSchemas

Proof. The “if” direction is obvious, since φ is a substitution instance of itself.

To prove the “only if” direction, we show the following: Suppose $\mathfrak{M} = \langle W, R, V \rangle$ is a modal model, and $\psi \equiv \varphi[\theta_1/p_1, \dots, \theta_n/p_n]$ is a substitution instance of φ . Define $\mathfrak{M}' = \langle W, R, V' \rangle$ by $V'(p_i) = \{w : \mathfrak{M}, w \Vdash \theta_i\}$. Then $\mathfrak{M}, w \Vdash \psi$ iff $\mathfrak{M}', w \Vdash \varphi$, for any $w \in W$. (We leave the proof as an exercise.) Now suppose that φ was valid, but some substitution instance ψ of φ was not valid. Then for some $\mathfrak{M} = \langle W, R, V \rangle$ and some $w \in W$, $\mathfrak{M}, w \not\Vdash \psi$. But then $\mathfrak{M}', w \not\Vdash \varphi$ by the claim, and φ is not valid, a contradiction. \square

Problem syn.2. Prove the claim in the “only if” part of the proof of **Proposition syn.6**. (Hint: use induction on φ .)

Note, however, that it is not true that a schema is true in a model iff its characteristic formula is. Of course, the “only if” direction holds: if every instance of φ is true in \mathfrak{M} , φ itself is true in \mathfrak{M} . But it may happen that φ is true in \mathfrak{M} but some instance of φ is false at some world in \mathfrak{M} . For a very simple counterexample consider p in a model with only one world w and $V(p) = \{w\}$, so that p is true at w . But \perp is an instance of p , and not true at w .

Problem syn.3. Show that none of the following **formulas** are valid:

- D: $\Box p \rightarrow \Diamond p$;
- T: $\Box p \rightarrow p$;
- B: $p \rightarrow \Box\Diamond p$;
- 4: $\Box p \rightarrow \Box\Box p$;
- 5: $\Diamond p \rightarrow \Box\Diamond p$.

Problem syn.4. Prove that the schemas in the first column of **Table 1** are valid and those in the second column are not valid.

Problem syn.5. Decide whether the following schemas are valid or invalid:

1. $(\Diamond\varphi \rightarrow \Box\psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$;

2. $\Diamond(\varphi \rightarrow \psi) \vee \Box(\psi \rightarrow \varphi)$.

Problem syn.6. For each of the following schemas find a model \mathfrak{M} such that every instance of the formula is true in \mathfrak{M} :

1. $p \rightarrow \Diamond\Diamond p$;

2. $\Diamond p \rightarrow \Box p$.

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Bibliography