Figure 1: Counterexample to \( p \rightarrow \lozenge p \models \square \neg p \rightarrow p \).

### syn.1 Entailment

With the definition of truth at a world, we can define an entailment relation between formulas. A formula \( \psi \) entails \( \varphi \) iff, whenever \( \psi \) is true, \( \varphi \) is true as well. Here, “whenever” means both “whichever model we consider” as well as “whichever world in that model we consider.”

**Definition syn.1.** If \( \Gamma \) is a set of formulas and \( \varphi \) a formula, then \( \Gamma \) entails \( \varphi \), in symbols: \( \Gamma \models \varphi \), if and only if for every model \( \mathcal{M} = \langle W, R, V \rangle \) and world \( w \in W \), if \( \mathcal{M}, w \models \psi \) for every \( \psi \in \Gamma \), then \( \mathcal{M}, w \models \varphi \). If \( \Gamma \) contains a single formula \( \psi \), then we write \( \psi \models \varphi \).

**Example syn.2.** To show that a formula entails another, we have to reason about all models, using the definition of \( \mathcal{M}, w \models \psi \). For instance, to show \( p \rightarrow \lozenge p \models \square \neg p \rightarrow \neg p \), we might argue as follows: Consider a model \( \mathcal{M} = \langle W, R, V \rangle \) and \( w \in W \), and suppose \( \mathcal{M}, w \models p \rightarrow \lozenge p \). We have to show that \( \mathcal{M}, w \not\models \square \neg p \rightarrow \neg p \). By assumption, \( \mathcal{M}, w \models p \rightarrow \lozenge p \) and \( \mathcal{M}, w \not\models \neg p \). Since \( \mathcal{M}, w \not\models \neg p \), \( \mathcal{M}, w \models p \). By definition of \( \mathcal{M}, w \models \varphi \), there is some \( w' \) with \( Rww' \) such that \( \mathcal{M}, w' \models p \). Since also \( \mathcal{M}, w \not\models \square \neg p \), \( \mathcal{M}, w' \not\models \neg p \), a contradiction.

To show that a formula \( \psi \) does not entail another \( \varphi \), we have to give a counterexample, i.e., a model \( \mathcal{M} = \langle W, R, V \rangle \) where we show that at some world \( w \in W \), \( \mathcal{M}, w \models \psi \) but \( \mathcal{M}, w \not\models \varphi \). Let’s show that \( p \rightarrow \lozenge p \not\models \square p \rightarrow p \). Consider the model in Figure 1. We have \( \mathcal{M}, w_1 \models \lozenge p \) and hence \( \mathcal{M}, w_1 \models p \rightarrow \lozenge p \). However, since \( \mathcal{M}, w_1 \not\models \square p \) but \( \mathcal{M}, w_1 \models p \), we have \( \mathcal{M}, w_1 \not\models \square p \rightarrow p \).

Often very simple counterexamples suffice. The model \( \mathcal{M}' = \{ W', R', V' \} \) with \( W' = \{ w \} \), \( R' = \emptyset \), and \( V'(p) = \emptyset \) is also a counterexample: Since \( \mathcal{M}', w \not\models p \), \( \mathcal{M}', w \models p \rightarrow \lozenge p \). As no worlds are accessible from \( w \), we have \( \mathcal{M}', w \not\models \square p \), and so \( \mathcal{M}', w \not\models \square p \rightarrow p \).

**Problem syn.1.** Show that \( \square (\varphi \land \psi) \models \square \varphi \).

**Problem syn.2.** Show that \( \square (p \rightarrow q) \not\models p \rightarrow \square q \) and \( p \rightarrow \square q \not\models \square (p \rightarrow q) \).