frd.1 Second-order Definability

Not every frame property definable by modal formulas is first-order definable. However, if we allow quantification over one-place predicates (i.e., monadic second-order quantification), we define all modally definable frame properties. The trick is to exploit a systematic way in which the conditions under which a modal formula is true at a world are related to first-order formulas. This is the so-called standard translation of modal formulas into first-order formulas in a language containing not just a two-place predicate symbol $Q$ for the accessibility relation, but also a one-place predicate symbol $P_i$ for the propositional variables $p_i$ occurring in $\phi$.

**Definition frd.1.** The *standard translation* $ST_x(\phi)$ is inductively defined as follows:

1. $\phi \equiv \bot$: $ST_x(\phi) = \bot$.
2. $\phi \equiv \top$: $ST_x(\phi) = \top$.
3. $\phi \equiv p_i$: $ST_x(\phi) = P_i(x)$.
4. $\phi \equiv \neg \psi$: $ST_x(\phi) = \neg ST_x(\psi)$.
5. $\phi \equiv (\psi \land \chi)$: $ST_x(\phi) = (ST_x(\psi) \land ST_x(\chi))$.
6. $\phi \equiv (\psi \lor \chi)$: $ST_x(\phi) = (ST_x(\psi) \lor ST_x(\chi))$.
7. $\phi \equiv (\psi \rightarrow \chi)$: $ST_x(\phi) = (ST_x(\psi) \rightarrow ST_x(\chi))$.
8. $\phi \equiv (\psi \leftrightarrow \chi)$: $ST_x(\phi) = (ST_x(\psi) \leftrightarrow ST_x(\chi))$.
9. $\phi \equiv \Box \psi$: $ST_x(\phi) = \forall y (Q(x, y) \rightarrow ST_y(\psi))$.
10. $\phi \equiv \Diamond \psi$: $ST_x(\phi) = \exists y (Q(x, y) \land ST_y(\psi))$.

For instance, $ST_x(\Box p \rightarrow p)$ is $\forall y (Q(x, y) \rightarrow P(y)) \rightarrow P(x)$. Any structure for the language of $ST_x(\phi)$ requires a domain, a two-place relation assigned to $Q$, and subsets of the domain assigned to the one-place predicate symbols $P_i$. In other words, the components of such a structure are exactly those of a model for $\phi$: the domain is the set of worlds, the two-place relation assigned to $Q$ is the accessibility relation, and the subsets assigned to $P_i$ are just the assignments $V(p_i)$. It won’t surprise that satisfaction of $\phi$ in a modal model and of $ST_x(\phi)$ in the corresponding structure agree:

**Proposition frd.2.** Let $\mathcal{M} = \langle W, R, V \rangle$, $\mathcal{M}'$ be the first-order structure with $|\mathcal{M}'| = W$, $Q^{\mathcal{M}'} = R$, and $P_i^{\mathcal{M}'} = V(p_i)$, and $s(x) = w$. Then

$\mathcal{M}, w \models \phi$ iff $\mathcal{M}', s \models ST_x(\phi)$

**Proof.** By induction on $\phi$. □
**Proposition frd.3.** Suppose \( \varphi \) is a modal formula and \( \mathcal{F} = \langle W, R \rangle \) is a frame. Let \( \mathcal{F}' \) be the first-order structure with \( |\mathcal{F}'| = W \) and \( Q^{\mathcal{F}'} = R \), and let \( \varphi' \) be the second-order formula

\[
\forall X_1 \ldots \forall X_n \forall x \ ST_x(\varphi)[X_1/P_1, \ldots, X_n/P_n],
\]

where \( P_1, \ldots, P_n \) are all one-place predicate symbols in \( ST_x(\varphi) \). Then

\[\mathcal{F} \models \varphi \iff \mathcal{F}' \models \varphi'.\]

**Proof.** \( \mathcal{F}' \models \varphi' \iff \) for every structure \( \mathcal{M}' \) where \( P_{i}^\mathcal{M}' \subseteq W \) for \( i = 1, \ldots, n \), and for every \( s \) with \( s(x) \in W \), \( \mathcal{M}, s \models ST_x(\varphi) \). By Proposition frd.2, that is the case iff for all models \( \mathcal{M} \) based on \( \mathcal{F} \) and every world \( w \in W \), \( \mathcal{M}, w \models \varphi \), i.e., \( \mathcal{F} \models \varphi \).

**Definition frd.4.** A class \( \mathcal{F} \) of frames is second-order definable if there is a sentence \( \varphi \) in the second-order language with a single two-place predicate symbol \( P \) and quantifiers only over monadic set variables such that \( \mathcal{F} = \langle W, R \rangle \in \mathcal{F} \iff \mathcal{M} \models \varphi \) in the structure \( \mathcal{M} \) with \( |\mathcal{M}| = W \) and \( P^\mathcal{M} = R \).

**Corollary frd.5.** If a class of frames is definable by a formula \( \varphi \), the corresponding class of accessibility relations is definable by a monadic second-order sentence.

**Proof.** The monadic second-order sentence \( \varphi' \) of the preceding proof has the required property.

As an example, consider again the formula \( \square p \rightarrow p \). It defines reflexivity. Reflexivity is of course first-order definable by the sentence \( \forall x \ Q(x, x) \). But it is also definable by the monadic second-order sentence

\[
\forall X \forall x (\forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x)).
\]

This means, of course, that the two sentences are equivalent. Here’s how you might convince yourself of this directly: First suppose the second-order sentence is true in a structure \( \mathcal{M} \). Since \( x \) and \( X \) are universally quantified, the remainder must hold for any \( x \in W \) and set \( X \subseteq W \), e.g., the set \( \{z : Rxz\} \) where \( R = Q^\mathcal{M} \). So, for any \( s \) with \( s(x) \in W \) and \( s(X) = \{z : Rxz\} \) we have \( \mathcal{M} \models \forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x) \). But by the way we’ve picked \( s(X) \) that means \( \mathcal{M}, s \models \forall y (Q(x, y) \rightarrow Q(x, y)) \rightarrow Q(x, x) \), which is equivalent to \( Q(x, x) \) since the antecedent is valid. Since \( s(x) \) is arbitrary, we have \( \mathcal{M} \models \forall x Q(x, x) \).

Now suppose that \( \mathcal{M} \models \forall x Q(x, x) \) and show that \( \mathcal{M} \models \forall X \forall x (\forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x)) \). Pick any assignment \( s \), and assume \( \mathcal{M}, s \models \forall y (Q(x, y) \rightarrow X(y)) \). Let \( s' \) be the \( y \)-variant of \( s \) with \( s'(y) = s(x) \); we have \( \mathcal{M}, s' \models Q(x, y) \rightarrow X(y) \), i.e., \( \mathcal{M}, s \models Q(x, x) \rightarrow X(x) \). Since \( \mathcal{M} \models \forall x Q(x, x) \), the antecedent is true, and we have \( \mathcal{M}, s \models X(x) \), which is what we needed to show.

Since some definable classes of frames are not first-order definable, not every monadic second-order sentence of the form \( \varphi' \) is equivalent to a first-order sentence. There is no effective method to decide which ones are.