frd.1 Second-order Definability

Not every frame property definable by modal formulas is first-order definable. However, if we allow quantification over one-place predicates (i.e., monadic second-order quantification), we define all modally definable frame properties. The trick is to exploit a systematic way in which the conditions under which a modal formula is true at a world are related to first-order formulas. This is the so-called standard translation of modal formulas into first-order formulas in a language containing not just a two-place predicate symbol $Q$ for the accessibility relation, but also a one-place predicate symbol $P_i$ for the propositional variables $p_i$ occurring in $\varphi$.

Definition frd.1. The standard translation $ST_x(\varphi)$ is inductively defined as follows:

1. $\varphi \equiv \bot$: $ST_x(\varphi) = \bot$.
2. $\varphi \equiv \top$: $ST_x(\varphi) = \top$.
3. $\varphi \equiv p_i$: $ST_x(\varphi) = P_i(x)$.
4. $\varphi \equiv \neg \psi$: $ST_x(\varphi) = \neg ST_y(\psi)$.
5. $\varphi \equiv (\psi \land \chi)$: $ST_x(\varphi) = (ST_x(\psi) \land ST_x(\chi))$.
6. $\varphi \equiv (\psi \lor \chi)$: $ST_x(\varphi) = (ST_x(\psi) \lor ST_x(\chi))$.
7. $\varphi \equiv (\psi \rightarrow \chi)$: $ST_x(\varphi) = (ST_x(\psi) \rightarrow ST_x(\chi))$.
8. $\varphi \equiv (\psi \leftrightarrow \chi)$: $ST_x(\varphi) = (ST_x(\psi) \leftrightarrow ST_x(\chi))$.
9. $\varphi \equiv \Box \psi$: $ST_x(\varphi) = \forall y (Q(x, y) \rightarrow ST_y(\psi))$.
10. $\varphi \equiv \Diamond \psi$: $ST_x(\varphi) = \exists y (Q(x, y) \land ST_y(\psi))$.

For instance, $ST_x(\Box p \rightarrow p)$ is $\forall y (Q(x, y) \rightarrow P(y)) \rightarrow P(x)$. Any structure for the language of $ST_x(\varphi)$ requires a domain, a two-place relation assigned to $Q$, and subsets of the domain assigned to the one-place predicate symbols $P_i$. In other words, the components of such a structure are exactly those of a model for $\varphi$: the domain is the set of worlds, the two-place relation assigned to $Q$ is the accessibility relation, and the subsets assigned to $P_i$ are just the assignments $V(p_i)$. It won’t surprise that satisfaction of $\varphi$ in a modal model and of $ST_x(\varphi)$ in the corresponding structure agree:

Proposition frd.2. Let $\mathcal{M} = \langle W, R, V \rangle$, $\mathcal{M}'$ be the first-order structure with $|\mathcal{M}'| = W$, $Q^{\mathcal{M}'} = R$, and $P_i^{\mathcal{M}'} = V(p_i)$, and $s(x) = w$. Then

$\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', s \models ST_x(\varphi)$

Proof. By induction on $\varphi$. □
Proposition frd.3. Suppose $\varphi$ is a modal formula and $\mathfrak{F} = \langle W, R \rangle$ is a frame. Let $\mathfrak{F}'$ be the first-order structure with $|\mathfrak{F}'| = W$ and $Q^{\mathfrak{F}'} = R$, and let $\varphi'$ be the second-order formula

$$\forall X_1 \ldots \forall X_n \forall x \ ST_x(\varphi)[X_1/P_1, \ldots, X_n/P_n],$$

where $P_1, \ldots, P_n$ are all one-place predicate symbols in $ST_x(\varphi)$. Then

$\mathfrak{F} \models \varphi$ iff $\mathfrak{F}' \models \varphi'$

Proof. $\mathfrak{F}' \models \varphi'$ iff for every structure $\mathfrak{M}'$ where $P_i^{\mathfrak{M}'} \subseteq W$ for $i = 1, \ldots, n$, and for every $s$ with $s(x) \in W$, $\mathfrak{M}', s \models ST_x(\varphi)$. By Proposition frd.2, that is the case iff for all models $\mathfrak{M}$ based on $\mathfrak{F}$ and every world $w \in W$, $\mathfrak{M}, w \models \varphi$, i.e., $\mathfrak{F} \models \varphi$.

Definition frd.4. A class $\mathcal{F}$ of frames is second-order definable if there is a sentence $\varphi$ in the second-order language with a single two-place predicate symbol $P$ and quantifiers only over monadic set variables such that $\mathfrak{F} = \langle W, R \rangle \in \mathcal{F}$ iff $\mathfrak{M} \models \varphi$ in the structure $\mathfrak{M}$ with $|\mathfrak{M}| = W$ and $P^{\mathfrak{M}} = R$.

Corollary frd.5. If a class of frames is definable by a formula $\varphi$, the corresponding class of accessibility relations is definable by a monadic second-order sentence.

Proof. The monadic second-order sentence $\varphi'$ of the preceding proof has the required property.

As an example, consider again the formula $\square p \rightarrow p$. It defines reflexivity. Reflexivity is of course first-order definable by the sentence $\forall x \ X(x, x)$. But it is also definable by the monadic second-order sentence

$$\forall X \forall x \ (\forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x)).$$

This means, of course, that the two sentences are equivalent. Here’s how you might convince yourself of this directly: First suppose the second-order sentence is true in a structure $\mathfrak{M}$. Since $x$ and $X$ are universally quantified, the remainder must hold for any $x \in W$ and set $X \subseteq W$; e.g., the set $\{z : Rxz\}$ where $R = Q^{\mathfrak{M}}$. So, for any $s$ with $s(x) \in W$ and $s(X) = \{z : Rxz\}$ we have $\mathfrak{M}, s \models \forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x)$. But by the way we’ve picked $s(X)$ that means $\mathfrak{M}, s \models \forall y (Q(x, y) \rightarrow Q(x, y)) \rightarrow Q(x, x)$, which is equivalent to $Q(x, x)$ since the antecedent is valid. Since $s(x)$ is arbitrary, we have $\mathfrak{M}, s \models \forall x Q(x, x)$.

Now suppose that $\mathfrak{M}, s \models \forall x Q(x, x)$ and show that $\mathfrak{M}, s \models \forall X \forall x (\forall y (Q(x, y) \rightarrow X(y)) \rightarrow X(x))$. Pick any assignment $s$, and assume $\mathfrak{M}, s \models \forall y (Q(x, y) \rightarrow X(y))$. Let $s'$ be the $y$-variant of $s$ with $s'(y) = s(x)$; we have $\mathfrak{M}, s' \models Q(x, y) \rightarrow X(y)$, i.e., $\mathfrak{M}, s \models Q(x, x) \rightarrow X(x)$. Since $\mathfrak{M}, s' \models Q(x, x)$, the antecedent is true, and we have $\mathfrak{M}, s \models X(x)$, which is what we needed to show.

Since some definable classes of frames are not first-order definable, not every monadic second-order sentence of the form $\varphi'$ is equivalent to a first-order sentence. There is no effective method to decide which ones are.

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Bibliography