

## frd.1 Equivalence Relations and S5

nml:frd:es5:  
sec The modal logic **S5** is characterized as the set of **formulas** valid on all universal frames, i.e., every world is accessible from every world, including itself. In such a scenario,  $\Box$  corresponds to necessity and  $\Diamond$  to possibility:  $\Box\varphi$  is true if  $\varphi$  is true at *every* world, and  $\Diamond\varphi$  is true if  $\varphi$  is true at *some* world. It turns out that **S5** can also be characterized as the **formulas** valid on all reflexive, symmetric, and transitive frames, i.e., on all *equivalence relations*.

**Definition frd.1.** A binary relation  $R$  on  $W$  is an *equivalence relation* if and only if it is reflexive, symmetric and transitive. A relation  $R$  on  $W$  is *universal* if and only if  $Ruv$  for all  $u, v \in W$ .

Since T, B, and 4 characterize the reflexive, symmetric, and transitive frames, the frames where the accessibility relation is an equivalence relation are exactly those in which all three **formulas** are valid. It turns out that the equivalence relations can also be characterized by other combinations of **formulas**, since the conditions with which we've defined equivalence relations are equivalent to combinations of other familiar conditions on  $R$ .

nml:frd:es5:  
prop:equivalences **Proposition frd.2.** *The following are equivalent:*

1.  $R$  is an equivalence relation;
2.  $R$  is reflexive and euclidean;
3.  $R$  is serial, symmetric, and euclidean;
4.  $R$  is serial, symmetric, and transitive.

*Proof.* Exercise. □

**Problem frd.1.** Prove **Proposition frd.2** by showing:

1. If  $R$  is symmetric and transitive, it is euclidean.
2. If  $R$  is reflexive, it is serial.
3. If  $R$  is reflexive and euclidean, it is symmetric.
4. If  $R$  is symmetric and euclidean, it is transitive.
5. If  $R$  is serial, symmetric, and transitive, it is reflexive.

Explain why this suffices for the proof that the conditions are equivalent.

**Proposition frd.2** is the semantic counterpart to **??**, in that it gives an equivalent characterization of the modal logic of frames over which  $R$  is an equivalence relation (the logic traditionally referred to as **S5**).

What is the relationship between universal and equivalence relations? Although every universal relation is an equivalence relation, clearly not every equivalence relation is universal. However, the **formulas** valid on all universal relations are exactly the same as those valid on all equivalence relations.

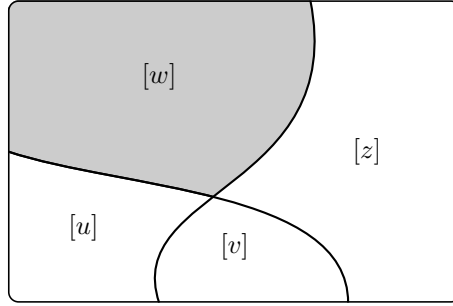


Figure 1: A partition of  $W$  in equivalence classes.

nml:frd:es5:  
fig:partition

**Proposition frd.3.** *Let  $R$  be an equivalence relation, and for each  $w \in W$  define the equivalence class of  $w$  as the set  $[w] = \{w' \in W : Rww'\}$ . Then:*

1.  $w \in [w]$ ;
2.  $R$  is universal on each equivalence class  $[w]$ ;
3. The collection of equivalence classes partitions  $W$  into mutually exclusive and jointly exhaustive subsets.

**Proposition frd.4.** *A formula  $\varphi$  is valid in all frames  $\mathfrak{F} = \langle W, R \rangle$  where  $R$  is an equivalence relation, if and only if it is valid in all frames  $\mathfrak{F} = \langle W, R \rangle$  where  $R$  is universal. Hence, the logic of universal frames is just **S5**.*

nml:frd:es5:  
prop:S5=univ

*Proof.* It's immediate to verify that a universal relation  $R$  on  $W$  is an equivalence. Hence, if  $\varphi$  is valid in all frames where  $R$  is an equivalence it is valid in all universal frames. For the other direction, we argue contrapositively: suppose  $\psi$  is a formula that fails at a world  $w$  in a model  $\mathfrak{M} = \langle W, R, V \rangle$  based on a frame  $\langle W, R \rangle$ , where  $R$  is an equivalence on  $W$ . So  $\mathfrak{M}, w \not\models \psi$ . Define a model  $\mathfrak{M}' = \langle W', R', V' \rangle$  as follows:

1.  $W' = [w]$ ;
2.  $R'$  is universal on  $W'$ ;
3.  $V'(p) = V(p) \cap W'$ .

(So the set  $W'$  of worlds in  $\mathfrak{M}'$  is represented by the shaded area in **Figure 1**.) It is easy to see that  $R$  and  $R'$  agree on  $W'$ . Then one can show by induction on formulas that for all  $w' \in W'$ :  $\mathfrak{M}', w' \models \varphi$  if and only if  $\mathfrak{M}, w' \models \varphi$  for each  $\varphi$  (this makes sense since  $W' \subseteq W$ ). In particular,  $\mathfrak{M}', w \not\models \psi$ , and  $\psi$  fails in a model based on a universal frame.  $\square$

**Photo Credits**

**Bibliography**