frd.1 Equivalence Relations and S5

nml:frd:es5:sec The modal logic **S5** is characterized as the set of formulas valid on all universal frames, i.e., every world is accessible from every world, including itself. In such a scenario, $\Box$ corresponds to necessity and $\Diamond$ to possibility: $\Box\varphi$ is true if φ is true at *every* world, and $\Diamond\varphi$ is true if φ is true at *some* world. It turns out that **S5** can also be characterized as the formulas valid on all reflexive, symmetric, and transitive frames, i.e., on all *equivalence relations*.

Definition frd.1. A binary relation R on W is an *equivalence relation* if and only if it is reflexive, symmetric and transitive. A relation R on W is *universal* if and only if Ruv for all $u, v \in W$.

Since T, B, and 4 characterize the reflexive, symmetric, and transitive frames, the frames where the accessibility relation is an equivalence relation are exactly those in which all three formulas are valid. It turns out that the equivalence relations can also be characterized by other combinations of formulas, since the conditions with which we've defined equivalence relations are equivalent to combinations of other familiar conditions on R.

nml:frd:es5:prop:equivalences **Proposition frd.2.** *The following are equivalent:*

1. *R is an equivalence relation;*
2. *R is reflexive and euclidean;*
3. *R is serial, symmetric, and euclidean;*
4. *R is serial, symmetric, and transitive.*

Proof. Exercise. $\Box$

Problem frd.1. Prove Proposition frd.2 by showing:

1. If R is symmetric and transitive, it is euclidean.
2. If R is reflexive, it is serial.
3. If R is reflexive and euclidean, it is symmetric.
4. If R is symmetric and euclidean, it is transitive.
5. If R is serial, symmetric, and transitive, it is reflexive.

Explain why this suffices for the proof that the conditions are equivalent.

Proposition frd.2 is the semantic counterpart to **??**, in that it gives an equivalent characterization of the modal logic of frames over which R is an equivalence relation (the logic traditionally referred to as **S5**).

What is the relationship between universal and equivalence relations? Although every universal relation is an equivalence relation, clearly not every equivalence relation is universal. However, the formulas valid on all universal relations are exactly the same as those valid on all equivalence relations.

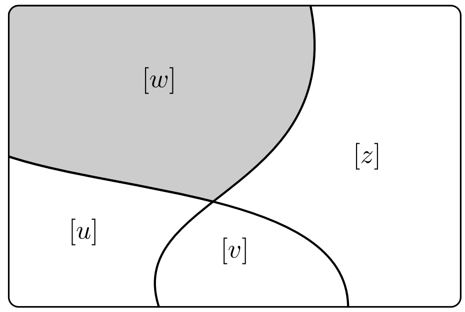

Figure 1: A partition of W in equivalence classes.

nml:frd:es5: fig:partition

Proposition frd.3. *Let R be an equivalence relation, and for each $w \in W$ define the* equivalence class *of w as the set $[w] = \{w' \in W : Rww'\}$. Then:*

1. *$w \in [w]$;*
2. *R is universal on each equivalence class $[w]$;*
3. *The collection of equivalence classes partitions W into mutually exclusive and jointly exhaustive subsets.*

Proposition frd.4. *A formula φ is valid in all frames $\mathfrak{F} = \langle W, R\rangle$ where R is an equivalence relation, if and only if it is valid in all frames $\mathfrak{F} = \langle W, R\rangle$ where R is universal. Hence, the logic of universal frames is just* **S5**. *nml:frd:es5: prop:S5=univ*

Proof. It's immediate to verify that a universal relation R on W is an equivalence. Hence, if φ is valid in all frames where R is an equivalence it is valid in all universal frames. For the other direction, we argue contrapositively: suppose ψ is a formula that fails at a world w in a model $\mathfrak{M} = \langle W, R, V\rangle$ based on a frame $\langle W, R\rangle$, where R is an equivalence on W. So $\mathfrak{M}, w \nVdash \psi$. Define a model $\mathfrak{M}' = \langle W', R', V'\rangle$ as follows:

1. $W' = [w]$;
2. R' is universal on W';
3. $V'(p) = V(p) \cap W'$.

(So the set W' of worlds in $\mathfrak{M}'$ is represented by the shaded area in Figure 1.) It is easy to see that R and R' agree on W'. Then one can show by induction on formulas that for all $w' \in W'$: $\mathfrak{M}', w' \Vdash \varphi$ if and only if $\mathfrak{M}, w' \Vdash \varphi$ for each φ (this makes sense since $W' \subseteq W$). In particular, $\mathfrak{M}', w \nVdash \psi$, and ψ fails in a model based on a universal frame. $\square$

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Bibliography