

## frd.1 Frame Definability

nml:frd:def:  
sec

Even though the converse implications of ?? fail, they hold if we replace “model” by “frame”: for the properties considered in ??, it is true that if a formula is valid in a frame then the accessibility relation of that frame has the corresponding property. So, the formulas considered define the classes of frames that have the corresponding property.

**Definition frd.1.** If  $\mathcal{F}$  is a class of frames, we say  $\varphi$  defines  $\mathcal{F}$  iff  $\mathfrak{F} \models \varphi$  for all and only frames  $\mathfrak{F} \in \mathcal{F}$ .

We now proceed to establish the full definability results for frames.

nml:frd:def:  
thm:fullCorrespondence

**Theorem frd.2.** If the formula on the right side of ?? is valid in a frame  $\mathfrak{F}$ , then  $\mathfrak{F}$  has the property on the left side.

- Proof.*
1. Suppose D is valid in  $\mathfrak{F} = \langle W, R \rangle$ , i.e.,  $\mathfrak{F} \models \Box p \rightarrow \Diamond p$ . Let  $\mathfrak{M} = \langle W, R, V \rangle$  be a model based on  $\mathfrak{F}$ , and  $w \in W$ . We have to show that there is a  $v$  such that  $Rwv$ . Suppose not: then both  $\mathfrak{M} \Vdash \Box \varphi$  and  $\mathfrak{M}, w \not\Vdash \Diamond \varphi$  for any  $\varphi$ , including  $p$ . But then  $\mathfrak{M}, w \not\Vdash \Box p \rightarrow \Diamond p$ , contradicting the assumption that  $\mathfrak{F} \models \Box p \rightarrow \Diamond p$ .
  2. Suppose T is valid in  $\mathfrak{F}$ , i.e.,  $\mathfrak{F} \models \Box p \rightarrow p$ . Let  $w \in W$  be an arbitrary world; we need to show  $Rww$ . Let  $u \in V(p)$  if and only if  $Rwu$  (when  $q$  is other than  $p$ ,  $V(q)$  is arbitrary, say  $V(q) = \emptyset$ ). Let  $\mathfrak{M} = \langle W, R, V \rangle$ . By construction, for all  $u$  such that  $Rwu$ :  $\mathfrak{M}, u \Vdash p$ , and hence  $\mathfrak{M}, w \Vdash \Box p$ . But by hypothesis  $\Box p \rightarrow p$  is true at  $w$ , so that  $\mathfrak{M}, w \Vdash p$ , but by definition of  $V$  this is possible only if  $Rww$ .
  3. We prove the contrapositive: Suppose  $\mathfrak{F}$  is not symmetric, we show that B, i.e.,  $p \rightarrow \Box \Diamond p$  is not valid in  $\mathfrak{F} = \langle W, R \rangle$ . If  $\mathfrak{F}$  is not symmetric, there are  $u, v \in W$  such that  $Ruv$  but not  $Rvu$ . Define  $V$  such that  $w \in V(p)$  if and only if not  $Rvw$  (and  $V$  is arbitrary otherwise). Let  $\mathfrak{M} = \langle W, R, V \rangle$ . Now, by definition of  $V$ ,  $\mathfrak{M}, w \Vdash p$  for all  $w$  such that not  $Rvw$ , in particular,  $\mathfrak{M}, u \Vdash p$  since not  $Rvu$ . Also, since  $Rvw$  iff  $w \notin V(p)$ , there is no  $w$  such that  $Rvw$  and  $\mathfrak{M}, w \Vdash p$ , and hence  $\mathfrak{M}, v \not\Vdash \Diamond p$ . Since  $Ruv$ , also  $\mathfrak{M}, u \not\Vdash \Box \Diamond p$ . It follows that  $\mathfrak{M}, u \not\Vdash p \rightarrow \Box \Diamond p$ , and so B is not valid in  $\mathfrak{F}$ .
  4. Suppose 4 is valid in  $\mathfrak{F} = \langle W, R \rangle$ , i.e.,  $\mathfrak{F} \models \Box p \rightarrow \Box \Box p$ , and let  $u, v, w \in W$  be arbitrary worlds such that  $Ruv$  and  $Rvw$ ; we need to show that  $Ruw$ . Define  $V$  such that  $z \in V(p)$  if and only if  $Ruz$  (and  $V$  is arbitrary otherwise). Let  $\mathfrak{M} = \langle W, R, V \rangle$ . By definition of  $V$ ,  $\mathfrak{M}, z \Vdash p$  for all  $z$  such that  $Ruz$ , and hence  $\mathfrak{M}, u \Vdash \Box p$ . But by hypothesis 4,  $\Box p \rightarrow \Box \Box p$ , is true at  $u$ , so that  $\mathfrak{M}, u \Vdash \Box \Box p$ . Since  $Ruv$  and  $Rvw$ , we have  $\mathfrak{M}, w \Vdash p$ , but by definition of  $V$  this is possible only if  $Ruw$ , as desired.

5. We proceed contrapositively, assuming that the frame  $\mathfrak{F} = \langle W, R \rangle$  is not euclidean, and show that it falsifies 5, i.e.,  $\mathfrak{F} \not\models \Diamond p \rightarrow \Box \Diamond p$ . Suppose there are worlds  $u, v, w \in W$  such that  $Rwu$  and  $Rwv$  but not  $Ruv$ . Define  $V$  such that for all worlds  $z$ ,  $z \in V(p)$  if and only if it is *not* the case that  $Ruz$ . Let  $\mathfrak{M} = \langle W, R, V \rangle$ . Then by hypothesis  $\mathfrak{M}, v \Vdash p$  and since  $Rwv$  also  $\mathfrak{M}, w \Vdash \Diamond p$ . However, there is no world  $y$  such that  $Ruy$  and  $\mathfrak{M}, y \Vdash p$  so  $\mathfrak{M}, u \not\models \Box \Diamond p$ . Since  $Rwu$ , it follows that  $\mathfrak{M}, w \not\models \Box \Diamond p$ , so that 5,  $\Diamond p \rightarrow \Box \Diamond p$ , fails at  $w$ .  $\square$

You'll notice a difference between the proof for D and the other cases: no mention was made of the valuation  $V$ . In effect, we proved that if  $\mathfrak{M} \Vdash D$  then  $\mathfrak{M}$  is serial. So D defines the class of serial *models*, not just frames.

**Corollary frd.3.** *Any model where D is true is serial.*

nml:frd:def:  
prop:D-serial

**Corollary frd.4.** *Each formula on the right side of ?? defines the class of frames which have the property on the left side.*

*Proof.* In ??, we proved that if a model has the property on the left, the formula on the right is true in it. Thus, if a frame  $\mathfrak{F}$  has the property on the left, the formula on the right is valid in  $\mathfrak{F}$ . In **Theorem frd.2**, we proved the converse implications: if a formula on the right is valid in  $\mathfrak{F}$ ,  $\mathfrak{F}$  has the property on the left.  $\square$

**Problem frd.1.** Show that if the formula on the right side of ?? is valid in a frame  $\mathfrak{F}$ , then  $\mathfrak{F}$  has the property on the left side. To do this, consider a frame that does *not* satisfy the property on the left, and define a suitable  $V$  such that the formula on the right is false at some world.

**Theorem frd.2** also shows that the properties can be combined: for instance if both B and 4 are valid in  $\mathfrak{F}$  then the frame is both symmetric and transitive, etc. Many important modal logics are characterized as the set of formulas valid in all frames that combine some frame properties, and so we can characterize them as the set of formulas valid in all frames in which the corresponding defining formulas are valid. For instance, the classical system **S4** is the set of all formulas valid in all reflexive and transitive frames, i.e., in all those where both T and 4 are valid. **S5** is the set of all formulas valid in all reflexive, symmetric, and euclidean frames, i.e., all those where all of T, B, and 5 are valid.

Logical relationships between properties of  $R$  in general correspond to relationships between the corresponding defining formulas. For instance, every reflexive relation is serial; hence, whenever T is valid in a frame, so is D. (Note that this relationship is *not* that of entailment. It is not the case that whenever  $\mathfrak{M}, w \Vdash T$  then  $\mathfrak{M}, w \Vdash D$ .) We record some such relationships.

**Proposition frd.5.** *Let  $R$  be a binary relation on a set  $W$ ; then:*

nml:frd:def:  
prop:relation-facts

1. *If  $R$  is reflexive, then it is serial.*

2. *If  $R$  is symmetric, then it is transitive if and only if it is euclidean.*
3. *If  $R$  is symmetric or euclidean then it is weakly directed (it has the “diamond property”).*
4. *If  $R$  is euclidean then it is weakly connected.*
5. *If  $R$  is functional then it is serial.*

**Problem frd.2.** Prove **Proposition frd.5**.

## Photo Credits

## Bibliography