

fil.1 Preliminaries

nml:fil:pre:sec Filtrations allow us to establish the decidability of our systems of modal logic by showing that they have the *finite model property*, i.e., that any **formula** that is true (false) in a model is also true (false) in a *finite* model. Filtrations are defined relative to sets of **formulas** which are closed under subformulas.

nml:fil:pre: defn:modallyclosed **Definition fil.1.** A set Γ of **formulas** is *closed under subformulas* if it contains every subformula of a **formula** in Γ . Further, Γ is *modally closed* if it is closed under subformulas and moreover $\varphi \in \Gamma$ implies $\Box\varphi, \Diamond\varphi \in \Gamma$.

For instance, given a **formula** φ , the set of all its sub-**formulas** is closed under sub-**formulas**. When we're defining a filtration of a model through the set of sub-**formulas** of φ , it will have the property we're after: it makes φ true (false) iff the original model does.

The set of worlds of a filtration of \mathfrak{M} through Γ is defined as the set of all equivalence classes of the following equivalence relation.

Definition fil.2. Let $\mathfrak{M} = \langle W, R, V \rangle$ and suppose Γ is closed under sub-**formulas**. Define a relation \equiv on W to hold of any two worlds that make the same **formulas** from Γ true, i.e.:

$$u \equiv v \quad \text{if and only if} \quad \forall \varphi \in \Gamma : \mathfrak{M}, u \Vdash \varphi \Leftrightarrow \mathfrak{M}, v \Vdash \varphi.$$

The equivalence class $[w]_{\equiv}$ of a world w , or $[w]$ for short, is the set of all worlds \equiv -equivalent to w :

$$[w] = \{v : v \equiv w\}.$$

Proposition fil.3. *Given \mathfrak{M} and Γ , \equiv as defined above is an equivalence relation, i.e., it is reflexive, symmetric, and transitive.*

Proof. The relation \equiv is reflexive, since w makes exactly the same **formulas** from Γ true as itself. It is symmetric since if u makes the same **formulas** from Γ true as v , the same holds for v and u . It is also transitive, since if u makes the same **formulas** from Γ true as v , and v as w , then u makes the same **formulas** from Γ true as w . \square

The relation \equiv , like any equivalence relation, divides W into *partitions*, i.e., subsets of W which are pairwise disjoint, and together cover all of W . Every $w \in W$ is an **element** of one of the partitions, namely of $[w]$, since $w \equiv w$. So the partitions $[w]$ cover all of W . They are pairwise disjoint, for if $u \in [w]$ and $u \in [v]$, then $u \equiv w$ and $u \equiv v$, and by symmetry and transitivity, $w \equiv v$, and so $[w] = [v]$.

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Bibliography