

## fl.1 Filtrations and Properties of Accessibility

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sec

As noted, filtrations of universal, serial, and reflexive models are always also universal, serial, or reflexive. But not every filtration of a symmetric or transitive model is symmetric or transitive, respectively. In some cases, however, it is possible to define filtrations so that this does hold. In order to do so, we proceed as in the definition of the coarsest filtration, but add additional conditions to the definition of  $R^*$ . Let  $\Gamma$  be closed under sub-formulas. Consider the relations  $C_i(u, v)$  in Table 1 between worlds  $u, v$  in a model  $\mathfrak{M} = \langle W, R, V \rangle$ . We can define  $R^*[u][v]$  on the basis of combinations of these conditions. For instance, if we stipulate that  $R^*[u][v]$  iff the condition  $C_1(u, v)$  holds, we get exactly the coarsest filtration. If we stipulate  $R^*[u][v]$  iff both  $C_1(u, v)$  and  $C_2(u, v)$  hold, we get a different filtration. It is “finer” than the coarsest since fewer pairs of worlds satisfy  $C_1(u, v)$  and  $C_2(u, v)$  than  $C_1(u, v)$  alone.

$C_1(u, v)$ :	if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \Vdash \Box\varphi$ then $\mathfrak{M}, v \Vdash \varphi$ ; and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \Vdash \varphi$ then $\mathfrak{M}, u \Vdash \Diamond\varphi$ ;
$C_2(u, v)$ :	if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, v \Vdash \Box\varphi$ then $\mathfrak{M}, u \Vdash \varphi$ ; and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, u \Vdash \varphi$ then $\mathfrak{M}, v \Vdash \Diamond\varphi$ ;
$C_3(u, v)$ :	if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, u \Vdash \Box\varphi$ then $\mathfrak{M}, v \Vdash \Box\varphi$ ; and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, v \Vdash \Diamond\varphi$ then $\mathfrak{M}, u \Vdash \Diamond\varphi$ ;
$C_4(u, v)$ :	if $\Box\varphi \in \Gamma$ and $\mathfrak{M}, v \Vdash \Box\varphi$ then $\mathfrak{M}, u \Vdash \Box\varphi$ ; and if $\Diamond\varphi \in \Gamma$ and $\mathfrak{M}, u \Vdash \Diamond\varphi$ then $\mathfrak{M}, v \Vdash \Diamond\varphi$ ;

Table 1: Conditions on possible worlds for defining filtrations.

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tab:Conditions  
thm:more-filtrations

**Theorem fl.1.** *Let  $\mathfrak{M} = \langle W, R, V \rangle$  be a model,  $\Gamma$  closed under sub-formulas. Let  $W^*$  and  $V^*$  be defined as in ???. Then:*

1. *Suppose  $R^*[u][v]$  if and only if  $C_1(u, v) \wedge C_2(u, v)$ . Then  $R^*$  is symmetric, and  $\mathfrak{M}^* = \langle W^*, R^*, V^* \rangle$  is a filtration if  $\mathfrak{M}$  is symmetric.*
2. *Suppose  $R^*[u][v]$  if and only if  $C_1(u, v) \wedge C_3(u, v)$ . Then  $R^*$  is transitive, and  $\mathfrak{M}^* = \langle W^*, R^*, V^* \rangle$  is a filtration if  $\mathfrak{M}$  is transitive.*
3. *Suppose  $R^*[u][v]$  if and only if  $C_1(u, v) \wedge C_2(u, v) \wedge C_3(u, v) \wedge C_4(u, v)$ . Then  $R^*$  is symmetric and transitive, and  $\mathfrak{M}^* = \langle W^*, R^*, V^* \rangle$  is a filtration if  $\mathfrak{M}$  is symmetric and transitive.*
4. *Suppose  $R^*$  is defined as  $R^*[u][v]$  if and only if  $C_1(u, v) \wedge C_3(u, v) \wedge C_4(u, v)$ . Then  $R^*$  is transitive and euclidean, and  $\mathfrak{M}^* = \langle W^*, R^*, V^* \rangle$  is a filtration if  $\mathfrak{M}$  is transitive and euclidean.*

*Proof.* 1. It’s immediate that  $R^*$  is symmetric, since  $C_1(u, v) \Leftrightarrow C_2(v, u)$  and  $C_2(u, v) \Leftrightarrow C_1(v, u)$ . So it’s left to show that if  $\mathfrak{M}$  is symmetric then  $\mathfrak{M}^*$  is a filtration through  $\Gamma$ . Condition  $C_1(u, v)$  guarantees that ?? and ?? of ??? are satisfied. So we just have to verify ?????, i.e., that  $Ruv$  implies  $R^*[u][v]$ .

So suppose  $Ruv$ . To show  $R^*[u][v]$  we need to establish that  $C_1(u, v)$  and  $C_2(u, v)$ . For  $C_1$ : if  $\Box\varphi \in \Gamma$  and  $\mathfrak{M}, u \Vdash \Box\varphi$  then also  $\mathfrak{M}, v \Vdash \varphi$  (since  $Ruv$ ). Similarly, if  $\Diamond\varphi \in \Gamma$  and  $\mathfrak{M}, v \Vdash \varphi$  then  $\mathfrak{M}, u \Vdash \Diamond\varphi$  since  $Ruv$ . For  $C_2$ : if  $\Box\varphi \in \Gamma$  and  $\mathfrak{M}, v \Vdash \Box\varphi$  then  $Ruv$  implies  $Rvu$  by symmetry, so that  $\mathfrak{M}, u \Vdash \varphi$ . Similarly, if  $\Diamond\varphi \in \Gamma$  and  $\mathfrak{M}, u \Vdash \varphi$  then  $\mathfrak{M}, v \Vdash \Diamond\varphi$  (since  $Rvu$  by symmetry).

2. Exercise.

3. Exercise.

4. Exercise. □

**Problem fl.1.** Complete the proof of **Theorem fl.1**.

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## Bibliography