

fil.1 Filtrations of Euclidean Models

nml:fil:eucl:sec The approach of ?? does not work in the case of models that are euclidean or serial and euclidean. Consider the model at the top of **Figure 1**, which is both euclidean and serial. Let $\Gamma = \{p, \Box p\}$. When taking a filtration through Γ , then $[w_1] = [w_3]$ since w_1 and w_3 are the only worlds that agree on Γ . Any filtration will also have the arrow inherited from \mathfrak{M} , as depicted in **Figure 2**. That model isn't euclidean. Moreover, we cannot add arrows to that model in order to make it euclidean. We would have to add double arrows between $[w_2]$ and $[w_4]$, and then also between w_2 and w_5 . But $\Box p$ is supposed to be true at w_2 , while p is false at w_5 .

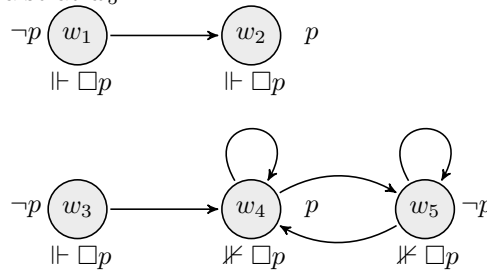


Figure 1: A serial and euclidean model.

nml:fil:eucl:fig:ser-eucl2

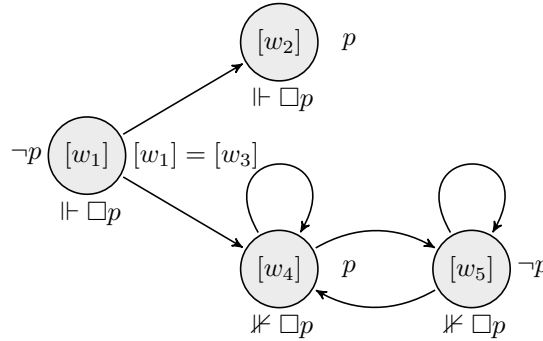


Figure 2: The filtration of the model in **Figure 1**.

nml:fil:eucl:fig:ser-eucl2 In particular, to obtain a euclidean filtration it is not enough to consider filtrations through arbitrary Γ 's closed under sub-formulas. Instead we need to consider sets Γ that are *modally closed* (see ??). Such sets of sentences are infinite, and therefore do not immediately yield a finite model property or the decidability of the corresponding system.

nml:fil:eucl:thm:modal-closed-filt **Theorem fil.1.** *Let Γ be modally closed, $\mathfrak{M} = \langle W, R, V \rangle$, and $\mathfrak{M}^* = \langle W^*, R^*, V^* \rangle$ be a coarsest filtration of \mathfrak{M} .*

1. *If \mathfrak{M} is symmetric, so is \mathfrak{M}^* .*
2. *If \mathfrak{M} is transitive, so is \mathfrak{M}^* .*
3. *If \mathfrak{M} is euclidean, so is \mathfrak{M}^* .*

- Proof.* 1. If \mathfrak{M}^* is a coarsest filtration, then by definition $R^*[u][v]$ holds if and only if $C_1(u, v)$. For transitivity, suppose $C_1(u, v)$ and $C_1(v, w)$; we have to show $C_1(u, w)$. Suppose $\mathfrak{M}, u \Vdash \Box\varphi$; then $\mathfrak{M}, u \Vdash \Box\Box\varphi$ since 4 is valid in all transitive models; since $\Box\Box\varphi \in \Gamma$ by closure, also by $C_1(u, v)$, $\mathfrak{M}, v \Vdash \Box\varphi$ and by $C_1(v, w)$, also $\mathfrak{M}, w \Vdash \varphi$. Suppose $\mathfrak{M}, w \Vdash \varphi$; then $\mathfrak{M}, v \Vdash \Diamond\varphi$ by $C_1(v, w)$, since $\Diamond\varphi \in \Gamma$ by modal closure. By $C_1(u, v)$, we get $\mathfrak{M}, u \Vdash \Diamond\Diamond\varphi$ since $\Diamond\Diamond\varphi \in \Gamma$ by modal closure. Since 4_\Diamond is valid in all transitive models, $\mathfrak{M}, u \Vdash \Diamond\varphi$.
2. Exercise. Use the fact that both 5 and 5_\Diamond are valid in all euclidean models.
3. Exercise. Use the fact that B and B_\Diamond are valid in all symmetric models. \square

Problem fil.1. Complete the proof of [Theorem fil.1](#).

Photo Credits

Bibliography