

fil.1 K and S5 have the Finite Model Property

nml:fil:fmp:
sec

Definition fil.1. A system Σ of modal logic is said to have the *finite model property* if whenever a **formula** φ is true at a world in a model of Σ then φ is true at a world in a *finite* model of Σ .

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prop:K-fmp

Proposition fil.2. **K** has the finite model property.

Proof. **K** is the set of valid **formulas**, i.e., any model is a model of **K**. By ??, if $\mathfrak{M}, w \Vdash \varphi$, then $\mathfrak{M}^*, w \Vdash \varphi$ for any filtration of \mathfrak{M} through the set Γ of sub-**formulas** of φ . Any **formula** only has finitely many sub-**formulas**, so Γ is finite. By ??, $|W^*| \leq 2^n$, where n is the number of **formulas** in Γ . And since **K** imposes no restriction on models, \mathfrak{M}^* is a **K**-model. \square

To show that a logic **L** has the finite model property via filtrations it is essential that the filtration of an **L**-model is itself a **L**-model. Often this requires a fair bit of work, and not any filtration yields a **L**-model. However, for universal models, this still holds.

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prop:univ-fin

Proposition fil.3. Let \mathcal{U} be the class of universal models (see ??) and \mathcal{U}_{Fin} the class of all finite universal models. Then any **formula** φ is valid in \mathcal{U} if and only if it is valid in \mathcal{U}_{Fin} .

Proof. Finite universal models are universal models, so the left-to-right direction is trivial. For the right-to-left direction, suppose that φ is false at some world w in a universal model \mathfrak{M} . Let Γ contain φ as well as all of its subformulas; clearly Γ is finite. Take a filtration \mathfrak{M}^* of \mathfrak{M} ; then \mathfrak{M}^* is finite by ??, and by ??, φ is false at $[w]$ in \mathfrak{M}^* . It remains to observe that \mathfrak{M}^* is also universal: given u and v , by hypothesis Ruv and by ????, also $R^*[u][v]$. \square

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cor:S5fmp

Corollary fil.4. **S5** has the finite model property.

Proof. By ??, if φ is true at a world in some reflexive and euclidean model then it is true at a world in a universal model. By **Proposition fil.3**, it is true at a world in a finite universal model (namely the filtration of the model through the set of sub-**formulas** of φ). Every universal model is also reflexive and euclidean; so φ is true at a world in a finite reflexive euclidean model. \square

Problem fil.1. Show that any filtration of a serial or reflexive model is also serial or reflexive (respectively).

Problem fil.2. Find a non-symmetric (non-transitive, non-euclidean) filtration of a symmetric (transitive, euclidean) model.

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Bibliography