

com.1 The Truth Lemma

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The canonical model \mathfrak{M}^Σ is defined in such a way that $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$ iff $\varphi \in \Delta$. For **propositional variables**, the definition of V^Σ yields this directly. We have to verify that the equivalence holds for all **formulas**, however. We do this by induction. The inductive step involves proving the equivalence for **formulas** involving propositional operators (where we have to use **??**) and the modal operators (where we invoke the results of **??**).

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Proposition com.1 (Truth Lemma). *For every formula φ , $\mathfrak{M}^\Sigma, \Delta \Vdash \varphi$ if and only if $\varphi \in \Delta$.*

Proof. By induction on φ .

1. $\varphi \equiv \perp$: $\mathfrak{M}^\Sigma, \Delta \not\Vdash \perp$ by **??**, and $\perp \notin \Delta$ by **????**.
2. $\varphi \equiv \top$: $\mathfrak{M}^\Sigma, \Delta \Vdash \top$ by **??**, and $\top \in \Delta$ by **????**.
3. $\varphi \equiv p$: $\mathfrak{M}^\Sigma, \Delta \Vdash p$ iff $\Delta \in V^\Sigma(p)$ by **??**. Also, $\Delta \in V^\Sigma(p)$ iff $p \in \Delta$ by definition of V^Σ .
4. $\varphi \equiv \neg\psi$: $\mathfrak{M}^\Sigma, \Delta \Vdash \neg\psi$ iff $\mathfrak{M}^\Sigma, \Delta \not\Vdash \psi$ (**??**) iff $\psi \notin \Delta$ (by inductive hypothesis) iff $\neg\psi \in \Delta$ (by **????**).
5. $\varphi \equiv \psi \wedge \chi$: $\mathfrak{M}^\Sigma, \Delta \Vdash \psi \wedge \chi$ iff $\mathfrak{M}^\Sigma, \Delta \Vdash \psi$ and $\mathfrak{M}^\Sigma, \Delta \Vdash \chi$ (by **??**) iff $\psi \in \Delta$ and $\chi \in \Delta$ (by inductive hypothesis) iff $\psi \wedge \chi \in \Delta$ (by **????**).
6. $\varphi \equiv \psi \vee \chi$: $\mathfrak{M}^\Sigma, \Delta \Vdash \psi \vee \chi$ iff $\mathfrak{M}^\Sigma, \Delta \Vdash \psi$ or $\mathfrak{M}^\Sigma, \Delta \Vdash \chi$ (by **??**) iff $\psi \in \Delta$ or $\chi \in \Delta$ (by inductive hypothesis) iff $\psi \vee \chi \in \Delta$ (by **????**).
7. $\varphi \equiv \psi \rightarrow \chi$: $\mathfrak{M}^\Sigma, \Delta \Vdash \psi \rightarrow \chi$ iff $\mathfrak{M}^\Sigma, \Delta \not\Vdash \psi$ or $\mathfrak{M}^\Sigma, \Delta \Vdash \chi$ (by **??**) iff $\psi \notin \Delta$ or $\chi \in \Delta$ (by inductive hypothesis) iff $\psi \rightarrow \chi \in \Delta$ (by **????**).
8. $\varphi \equiv \psi \leftrightarrow \chi$: $\mathfrak{M}^\Sigma, \Delta \Vdash \psi \leftrightarrow \chi$ iff either $\mathfrak{M}^\Sigma, \Delta \Vdash \psi$ and $\mathfrak{M}^\Sigma, \Delta \Vdash \chi$ or $\mathfrak{M}^\Sigma, \Delta \not\Vdash \psi$ and $\mathfrak{M}^\Sigma, \Delta \not\Vdash \chi$ (by **??**) iff either $\psi \in \Delta$ and $\chi \in \Delta$ or $\psi \notin \Delta$ and $\chi \notin \Delta$ (by inductive hypothesis) iff $\psi \leftrightarrow \chi \in \Delta$ (by **????**).
9. $\varphi \equiv \Box\psi$: First suppose that $\mathfrak{M}^\Sigma, \Delta \Vdash \Box\psi$. By **??**, for every Δ' such that $R^\Sigma \Delta \Delta'$, $\mathfrak{M}^\Sigma, \Delta' \Vdash \psi$. By inductive hypothesis, for every Δ' such that $R^\Sigma \Delta \Delta'$, $\psi \in \Delta'$. By definition of R^Σ , for every Δ' such that $\Box^{-1}\Delta \subseteq \Delta'$, $\psi \in \Delta'$. By **??**, $\Box\psi \in \Delta$.
Now assume $\Box\psi \in \Delta$. Let $\Delta' \in W^\Sigma$ be such that $R^\Sigma \Delta \Delta'$, i.e., $\Box^{-1}\Delta \subseteq \Delta'$. Since $\Box\psi \in \Delta$, $\psi \in \Box^{-1}\Delta$. Consequently, $\psi \in \Delta'$. By inductive hypothesis, $\mathfrak{M}^\Sigma, \Delta' \Vdash \psi$. Since Δ' is arbitrary with $R^\Sigma \Delta \Delta'$, for all $\Delta' \in W^\Sigma$ such that $R^\Sigma \Delta \Delta'$, $\mathfrak{M}^\Sigma, \Delta' \Vdash \psi$. By **??**, $\mathfrak{M}^\Sigma, \Delta \Vdash \Box\psi$.
10. $\varphi \equiv \Diamond\psi$: First suppose that $\mathfrak{M}^\Sigma, \Delta \Vdash \Diamond\psi$. By **??**, for some Δ' such that $R^\Sigma \Delta \Delta'$, $\mathfrak{M}^\Sigma, \Delta' \Vdash \psi$. By inductive hypothesis, for some Δ' such that $R^\Sigma \Delta \Delta'$, $\psi \in \Delta'$. By definition of R^Σ , for some Δ' such that $\Box^{-1}\Delta \subseteq \Delta'$,

$\psi \in \Delta'$. By ??, for some Δ' such that $\diamond\Delta' \subseteq \Delta$, $\psi \in \Delta'$. Since $\psi \in \Delta'$, $\diamond\psi \in \diamond\Delta'$, so $\diamond\psi \in \Delta$.

Now assume $\diamond\psi \in \Delta$. By ??, there is a complete Σ -consistent $\Delta' \in W^\Sigma$ such that $\diamond\Delta' \subseteq \Delta$ and $\psi \in \Delta'$. By ??, there is a $\Delta' \in W^\Sigma$ such that $\square^{-1}\Delta \subseteq \Delta'$, and $\psi \in \Delta'$. By definition of R^Σ , $R^\Sigma\Delta\Delta'$, so there is a $\Delta' \in W^\Sigma$ such that $R^\Sigma\Delta\Delta'$ and $\psi \in \Delta'$. By ??, $\mathfrak{M}^\Sigma, \Delta \Vdash \diamond\psi$. \square

Problem com.1. Complete the proof of [Proposition com.1](#).

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