com.1 The Truth Lemma

The canonical model $M^Σ$ is defined in such a way that $M^Σ, Δ \vDash \varphi$ if $\varphi \in Δ$. For propositional variables, the definition of $V^Σ$ yields this directly. We have to verify that the equivalence holds for all formulas, however. We do this by induction. The inductive step involves proving the equivalence for formulas involving propositional operators (where we have to use ??) and the modal operators (where we invoke the results of ??).

Proposition com.1 (Truth Lemma). For every formula $\varphi$, $M^Σ, Δ \vDash \varphi$ if and only if $\varphi \in Δ$.

Proof. By induction on $\varphi$.

1. $\varphi \equiv \bot$: $M^Σ, Δ \not\vDash \bot$ by ??, and $\bot \notin Δ$ by ???.
2. $\varphi \equiv \top$: $M^Σ, Δ \vDash \top$ by ??, and $\top \in Δ$ by ???.
3. $\varphi \equiv p$: $M^Σ, Δ \vDash p$ if $Δ \in V^Σ(p)$ by ?? Also, $Δ \in V^Σ(p)$ if $p \in Δ$ by definition of $V^Σ$.
4. $\varphi \equiv \neg ψ$: $M^Σ, Δ \vDash \neg ψ$ if $M^Σ, Δ \not\vDash ψ$ (by ??) if $ψ \notin Δ$ (by inductive hypothesis) if $\neg ψ \in Δ$ (by ???).
5. $\varphi \equiv ψ \land χ$: $M^Σ, Δ \vDash ψ \land χ$ if $M^Σ, Δ \vDash ψ$ and $M^Σ, Δ \vDash χ$ (by ??) if $ψ \in Δ$ and $χ \in Δ$ (by inductive hypothesis) if $ψ \land χ \in Δ$ (by ???).
6. $\varphi \equiv ψ \lor χ$: $M^Σ, Δ \vDash ψ \lor χ$ if $M^Σ, Δ \vDash ψ$ or $M^Σ, Δ \vDash χ$ (by ??) if $ψ \in Δ$ or $χ \in Δ$ (by inductive hypothesis) if $ψ \lor χ \in Δ$ (by ???).
7. $\varphi \equiv ψ \rightarrow χ$: $M^Σ, Δ \vDash ψ \rightarrow χ$ if $M^Σ, Δ \not\vDash ψ$ or $M^Σ, Δ \vDash χ$ (by ??) if $ψ \notin Δ$ or $χ \in Δ$ (by inductive hypothesis) if $ψ \rightarrow χ \in Δ$ (by ???).
8. $\varphi \equiv ψ \leftarrow χ$: $M^Σ, Δ \vDash ψ \leftarrow χ$ if either $M^Σ, Δ \not\vDash ψ$ and $M^Σ, Δ \vDash χ$ or $M^Σ, Δ \not\vDash χ$ and $M^Σ, Δ \vDash χ$ (by ??) if either $ψ \notin Δ$ and $χ \in Δ$ or $ψ \in Δ$ and $χ \notin Δ$ (by inductive hypothesis) if $ψ \leftarrow χ \in Δ$ (by ???).
9. $\varphi \equiv \Box ψ$: First suppose that $M^Σ, Δ \vDash \Box ψ$. By ??, for every $Δ'$ such that $R^ΣΔΔ', M^Σ, Δ' \vDash ψ$. By inductive hypothesis, for every $Δ'$ such that $R^ΣΔΔ', ψ \in Δ'$. By definition of $R^Σ$, for every $Δ'$ such that $\Box^{-1} Δ \subseteq Δ'$, $ψ \in Δ'$. By ??, $\Box ψ \in Δ$.

Now assume $\Box ψ \in Δ$. Let $Δ' \in W^Σ$ be such that $R^ΣΔΔ'$, i.e., $\Box^{-1} Δ \subseteq Δ'$. Since $\Box ψ \in Δ$, $ψ \in \Box^{-1} Δ$. Consequently, $ψ \in Δ'$. By inductive hypothesis, $M^Σ, Δ' \vDash ψ$. Since $Δ'$ is arbitrary with $R^ΣΔΔ'$, for all $Δ' \in W^Σ$ such that $R^ΣΔΔ'$, $M^Σ, Δ' \vDash ψ$. By ??, $M^Σ, Δ \vDash \Box ψ$.

10. $\varphi \equiv ◊ ψ$: First suppose that $M^Σ, Δ \vDash ◊ ψ$. By ??, for some $Δ'$ such that $R^ΣΔΔ', M^Σ, Δ' \vDash ψ$. By inductive hypothesis, for some $Δ'$ such that $R^ΣΔΔ', ψ \in Δ'$. By definition of $R^Σ$, for some $Δ'$ such that $\Box^{-1} Δ \subseteq Δ'$.
ψ ∈ Δ'. By ??, for some Δ' such that ◊Δ' ⊆ Δ, ψ ∈ Δ'. Since ψ ∈ Δ',
◊ψ ∈ ◊Δ', so ◊ψ ∈ Δ.

Now assume ◊ψ ∈ Δ. By ??, there is a complete Σ-consistent Δ' ∈ W^Σ such that ◊Δ' ⊆ Δ and ψ ∈ Δ'. By ??, there is a Δ' ∈ W^Σ such that
□^{-1}Δ ⊆ Δ', and ψ ∈ Δ'. By definition of R^Σ, R^ΣΔΔ', so there is a
Δ' ∈ W^Σ such that R^ΣΔΔ' and ψ ∈ Δ'. By ??, \(\mathfrak{M}^\Sigma, Δ \models ◊ψ\).

Problem com.1. Complete the proof of Proposition com.1.

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Bibliography