Modalities and Complete Consistent Sets

When we construct a model $M^\Sigma$ whose set of worlds is given by the complete $\Sigma$-consistent sets $\Delta$ in some normal modal logic $\Sigma$, we will also need to define an accessibility relation $R^\Sigma$ between such “worlds.” We want it to be the case that the accessibility relation (and the assignment $V^\Sigma$) are defined in such a way that $M^\Sigma, \Delta \models \varphi$ iff $\varphi \in \Delta$. How should we do this?

Once the accessibility relation is defined, the definition of truth at a world ensures that $M^\Sigma, \Delta \models \Box \varphi$ iff $M^\Sigma, \Delta' \models \varphi$ for all $\Delta'$ such that $R^\Sigma \Delta \Delta'$. The proof that $M^\Sigma, \Delta \models \varphi$ iff $\varphi \in \Delta$ requires that this is true in particular for formulas starting with a modal operator, i.e., $M^\Sigma, \Delta \models \Box \varphi$ iff $\Box \varphi \in \Delta$. Combining this requirement with the definition of truth at a world for $\Box \varphi$ yields:

$\Box \varphi \in \Delta$ iff $\varphi \in \Delta'$ for all $\Delta'$ with $R^\Sigma \Delta \Delta'$.

Consider the left-to-right direction: it says that if $\Box \varphi \in \Delta$, then $\varphi \in \Delta'$ for any $\varphi$ and any $\Delta'$ with $R^\Sigma \Delta \Delta'$. If we stipulate that $R^\Sigma \Delta \Delta'$ iff $\Box \varphi \in \Delta$, then this holds. We can write the condition on the right of the “iff” more compactly as: $\{ \varphi : \Box \varphi \in \Delta \} \subseteq \Delta'$.

So the question is: does this definition of $R^\Sigma$ in fact guarantee that $\Box \varphi \in \Delta$ iff $M^\Sigma, \Delta \models \Box \varphi$? Does it also guarantee that $\Diamond \varphi \in \Delta$ iff $M^\Sigma, \Delta \models \Diamond \varphi$? The next few results will establish this.

**Definition com.1.** If $\Gamma$ is a set of formulas, let

$$\Box \Gamma = \{ \Box \psi : \psi \in \Gamma \}$$
$$\Diamond \Gamma = \{ \Diamond \psi : \psi \in \Gamma \}$$

and

$$\Box^{-1} \Gamma = \{ \psi : \Box \psi \in \Gamma \}$$
$$\Diamond^{-1} \Gamma = \{ \psi : \Diamond \psi \in \Gamma \}$$

In other words, $\Box \Gamma$ is $\Gamma$ with $\Box$ in front of every formula in $\Gamma$; $\Box^{-1} \Gamma$ is all the $\Box$ed formulas of $\Gamma$ with the initial $\Box$’s removed. This definition is not terribly important on its own, but will simplify the notation considerably.

Note that $\Box \Box^{-1} \Gamma \subseteq \Gamma$:

$$\Box \Box^{-1} \Gamma = \{ \Box \psi : \Box \psi \in \Gamma \}$$

i.e., it’s just the set of all those formulas of $\Gamma$ that start with $\Box$.

**Lemma com.2.** If $\Gamma \models_\Sigma \varphi$ then $\Box \Gamma \models_\Sigma \Box \varphi$.

**Proof.** If $\Gamma \models_\Sigma \varphi$ then there are $\psi_1, \ldots, \psi_k \in \Gamma$ such that $\Sigma \vdash \psi_1 \to (\psi_2 \to \cdots (\psi_n \to \varphi) \cdots)$. Since $\Sigma$ is normal, by rule RK, $\Sigma \vdash \Box \psi_1 \to (\Box \psi_2 \to \cdots (\Box \psi_n \to \Box \varphi) \cdots)$, where obviously $\Box \psi_1, \ldots, \Box \psi_k \in \Box \Gamma$. Hence, by definition, $\Box \Gamma \models_\Sigma \Box \varphi$.

$\Box$
Lemma com.3. If $\Box^{-1} \Gamma \vdash \Sigma \varphi$ then $\Gamma \vdash \Box \varphi$. 

Proof. Suppose $\Box^{-1} \Gamma \vdash \Sigma \varphi$; then by Lemma com.2, $\Box \Box^{-1} \Gamma \vdash \Box \varphi$. But since $\Box \Box^{-1} \Gamma \subseteq \Gamma$, also $\Gamma \vdash \Box \varphi$ by Monotony. 

Proposition com.4. If $\Gamma$ is complete $\Sigma$-consistent, then $\Box \varphi \in \Gamma$ if and only if for every complete $\Sigma$-consistent $\Delta$ such that $\Box^{-1} \Gamma \subseteq \Delta$, it holds that $\varphi \in \Delta$. 

Proof. Suppose $\Gamma$ is complete $\Sigma$-consistent. The “only if” direction is easy: Suppose $\Box \varphi \in \Gamma$ and that $\Box^{-1} \Gamma \subseteq \Delta$. Since $\Box \varphi \in \Gamma$, $\varphi \in \Box^{-1} \Gamma \subseteq \Delta$, so $\varphi \in \Delta$.

For the “if” direction, we prove the contrapositive: Suppose $\Box \varphi \notin \Gamma$. Since $\Gamma$ is complete $\Sigma$-consistent, it is deductively closed, and hence $\Gamma \not\vdash \Box \varphi$. By Lemma com.3, $\Box^{-1} \Gamma \not\vdash \varphi$. By ????, $\Box^{-1} \Gamma \cup \{\neg \varphi\}$ is $\Sigma$-consistent. By Lindenbaum’s Lemma, there is a complete $\Sigma$-consistent set $\Delta$ such that $\Box^{-1} \Gamma \cup \{\neg \varphi\} \subseteq \Delta$. By consistency, $\varphi \notin \Delta$. 

Lemma com.5. Suppose $\Gamma$ and $\Delta$ are complete $\Sigma$-consistent. Then: $\Box^{-1} \Gamma \subseteq \Delta$ if and only if $\Diamond \Sigma \subseteq \Gamma$.

Proof. “Only if” direction: Assume $\Box^{-1} \Gamma \subseteq \Delta$ and suppose $\Diamond \varphi \in \Diamond \Delta$ (i.e., $\varphi \in \Delta$). In order to show $\Diamond \varphi \in \Gamma$ it suffices to show $\Box \neg \varphi \notin \Gamma$ for then by maximality $\Box \neg \varphi \in \Gamma$. Now, if $\Box \neg \varphi \in \Gamma$ then by hypothesis $\neg \varphi \in \Delta$, against the consistency of $\Delta$ (since $\varphi \in \Delta$). Hence $\Box \neg \varphi \notin \Gamma$, as required.

“If” direction: Assume $\Diamond \Delta \subseteq \Gamma$. We argue contrapositively: suppose $\varphi \notin \Delta$ in order to show $\Box \varphi \notin \Gamma$. If $\varphi \notin \Delta$ then by maximality $\neg \varphi \in \Delta$ and so by hypothesis $\Diamond \neg \varphi \in \Gamma$. But in a normal modal logic $\Diamond \neg \varphi$ is equivalent to $\neg \Box \varphi$, and if the latter is in $\Gamma$, by consistency $\Box \varphi \notin \Gamma$, as required.

Proposition com.6. If $\Gamma$ is complete $\Sigma$-consistent, then $\Diamond \varphi \in \Gamma$ if and only if for some complete $\Sigma$-consistent $\Delta$ such that $\Diamond \Sigma \subseteq \Gamma$, it holds that $\varphi \in \Delta$.

Proof. Suppose $\Gamma$ is complete $\Sigma$-consistent. $\Diamond \varphi \in \Gamma$ iff $\neg \Box \neg \varphi \in \Gamma$ by dual and closure. $\neg \Box \neg \varphi \in \Gamma$ iff $\Box \neg \varphi \notin \Gamma$ by ????, since $\Gamma$ is complete $\Sigma$-consistent. By Proposition com.4, $\Box \neg \varphi \notin \Gamma$ iff, for some complete $\Sigma$-consistent $\Delta$ with $\Box^{-1} \Gamma \subseteq \Delta$, $\neg \varphi \notin \Delta$. Now consider any such $\Delta$. By Lemma com.5, $\Box^{-1} \Gamma \subseteq \Delta$ iff $\Diamond \Delta \subseteq \Gamma$. Also, $\neg \varphi \notin \Delta$ iff $\varphi \in \Delta$ by ????. So $\Diamond \varphi \in \Gamma$ iff, for some complete $\Sigma$-consistent $\Delta$ with $\Diamond \Delta \subseteq \Gamma$, $\varphi \in \Delta$.

Problem com.1. Show that if $\Gamma$ is complete $\Sigma$-consistent, then $\Diamond \varphi \in \Gamma$ if and only if there is a complete $\Sigma$-consistent $\Delta$ such that $\Box^{-1} \Gamma \subseteq \Delta$ and $\varphi \in \Delta$. Do this without using Lemma com.5.