

## com.1 Lindenbaum's Lemma

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Lindenbaum's Lemma establishes that every  $\Sigma$ -consistent set of formulas is contained in at least one complete  $\Sigma$ -consistent set. Our construction of the canonical model will show that for each complete  $\Sigma$ -consistent set  $\Delta$ , there is a world in the canonical model where all and only the formulas in  $\Delta$  are true. So Lindenbaum's Lemma guarantees that every  $\Sigma$ -consistent set is true at some world in the canonical model.

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**Theorem com.1 (Lindenbaum's Lemma).** *If  $\Gamma$  is  $\Sigma$ -consistent then there is a complete  $\Sigma$ -consistent set  $\Delta$  extending  $\Gamma$ .*

*Proof.* Let  $\varphi_0, \varphi_1, \dots$  be an exhaustive listing of all formulas of the language (repetitions are allowed). For instance, start by listing  $\rho_0$ , and at each stage  $n \geq 1$  list the finitely many formulas of length  $n$  using only variables among  $\rho_0, \dots, \rho_n$ . We define sets of formulas  $\Delta_n$  by induction on  $n$ , and we then set  $\Delta = \bigcup_n \Delta_n$ . We first put  $\Delta_0 = \Gamma$ . Supposing that  $\Delta_n$  has been defined, we define  $\Delta_{n+1}$  by:

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\varphi_n\}, & \text{if } \Delta_n \cup \{\varphi_n\} \text{ is consistent;} \\ \Delta_n \cup \{\neg\varphi_n\}, & \text{otherwise.} \end{cases}$$

If we now let  $\Delta = \bigcup_{n=0}^{\infty} \Delta_n$ .

We have to show that this definition actually yields a set  $\Delta$  with the required properties, i.e.,  $\Gamma \subseteq \Delta$  and  $\Delta$  is complete  $\Sigma$ -consistent.

It's obvious that  $\Gamma \subseteq \Delta$ , since  $\Delta_0 \subseteq \Delta$  by construction, and  $\Delta_0 = \Gamma$ . In fact,  $\Delta_n \subseteq \Delta$  for all  $n$ , since  $\Delta$  is the union of all  $\Delta_n$ . (Since in each step of the construction, we add a formula to the set already constructed,  $\Delta_n \subseteq \Delta_{n+1}$ , so since  $\subseteq$  is transitive,  $\Delta_n \subseteq \Delta_m$  whenever  $n \leq m$ .) At each stage of the construction, we either add  $\varphi_n$  or  $\neg\varphi_n$ , and every formula appears (at least once) in the list of all  $\varphi_n$ . So, for every  $\varphi$  either  $\varphi \in \Delta$  or  $\neg\varphi \in \Delta$ , so  $\Delta$  is complete by definition.

Finally, we have to show, that  $\Delta$  is  $\Sigma$ -consistent. To do this, we show that (a) if  $\Delta$  were  $\Sigma$ -inconsistent, then some  $\Delta_n$  would be  $\Sigma$ -inconsistent, and (b) all  $\Delta_n$  are  $\Sigma$ -consistent.

So suppose  $\Delta$  were  $\Sigma$ -inconsistent. Then  $\Delta \vdash_{\Sigma} \perp$ , i.e., there are  $\varphi_1, \dots, \varphi_k \in \Delta$  such that  $\Sigma \vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow \dots (\varphi_k \rightarrow \perp) \dots)$ . Since  $\Delta = \bigcap_{n=0}^{\infty} \Delta_n$ , each  $\varphi_i \in \Delta_{n_i}$  for some  $n_i$ . Let  $n$  be the largest of these. Since  $n_i \leq n$ ,  $\Delta_{n_i} \subseteq \Delta_n$ . So, all  $\varphi_i$  are in some  $\Delta_n$ . This would mean  $\Delta_n \vdash_{\Sigma} \perp$ , i.e.,  $\Delta_n$  is  $\Sigma$ -inconsistent.

To show that each  $\Delta_n$  is  $\Sigma$ -consistent, we use a simple induction on  $n$ .  $\Delta_0 = \Gamma$ , and we assumed  $\Gamma$  was  $\Sigma$ -consistent. So the claim holds for  $n = 0$ . Now suppose it holds for  $n$ , i.e.,  $\Delta_n$  is  $\Sigma$ -consistent.  $\Delta_{n+1}$  is either  $\Delta_n \cup \{\varphi_n\}$  or  $\Delta_n \cup \{\neg\varphi_n\}$ . In the first case,  $\Delta_{n+1}$  is clearly  $\Sigma$ -consistent. However, by ????, either  $\Delta_n \cup \{\varphi_n\}$  or  $\Delta_n \cup \{\neg\varphi_n\}$  is consistent, so  $\Delta_{n+1}$  is consistent in the other case as well.  $\square$

**Corollary com.2.**  $\Gamma \vdash_{\Sigma} \varphi$  if and only if  $\varphi \in \Delta$  for each complete  $\Sigma$ -consistent set  $\Delta$  extending  $\Gamma$  (including when  $\Gamma = \emptyset$ , in which case we get another characterization of the modal system  $\Sigma$ .) [nml:com:lin:](#)  
[cor:provability-characterization](#)

*Proof.* Suppose  $\Gamma \vdash_{\Sigma} \varphi$ , and let  $\Delta$  be any complete  $\Sigma$ -consistent set extending  $\Gamma$ . If  $\varphi \notin \Delta$  then by maximality  $\neg\varphi \in \Delta$  and so  $\Delta \vdash_{\Sigma} \varphi$  (by monotony) and  $\Delta \vdash_{\Sigma} \neg\varphi$  (by reflexivity), and so  $\Delta$  is inconsistent. Conversely if  $\Gamma \not\vdash_{\Sigma} \varphi$ , then  $\Gamma \cup \{\neg\varphi\}$  is  $\Sigma$ -consistent, and by Lindenbaum's Lemma there is a complete consistent set  $\Delta$  extending  $\Gamma \cup \{\neg\varphi\}$ . By consistency,  $\varphi \notin \Delta$ .  $\square$

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## Bibliography