com.1 Introduction

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If Σ is a modal system, then the soundness theorem establishes that if $\Sigma \vdash \varphi$, then φ is valid in any class $\mathcal{C}$ of models in which all instances of all formulas in Σ are valid. In particular that means that if $\mathbf{K} \vdash \varphi$ then φ is true in all models; if $\mathbf{KT} \vdash \varphi$ then φ is true in all reflexive models; if $\mathbf{KD} \vdash \varphi$ then φ is true in all serial models, etc.

Completeness is the converse of soundness: that $\mathbf{K}$ is complete means that if a formula φ is valid, $\vdash \varphi$, for instance. Proving completeness is a lot harder to do than proving soundness. It is useful, first, to consider the contrapositive: $\mathbf{K}$ is complete iff whenever $\nvdash \varphi$, there is a countermodel, i.e., a model $\mathfrak{M}$ such that $\mathfrak{M} \nVdash \varphi$. Equivalently (negating φ), we could prove that whenever $\nvdash \neg\varphi$, there is a model of φ. In the construction of such a model, we can use information contained in φ. When we find models for specific formulas we often do the same: E.g., if we want to find a countermodel to $p \to \Box q$, we know that it has to contain a world where p is true and $\Box q$ is false. And a world where $\Box q$ is false means there has to be a world accessible from it where q is false. And that's all we need to know: which worlds make the propositional variables true, and which worlds are accessible from which worlds.

In the case of proving completeness, however, we don't have a specific formula φ for which we are constructing a model. We want to establish that a model exists for every φ such that $\nvdash_\Sigma \neg\varphi$. This is a minimal requirement, since *if* $\vdash_\Sigma \neg\varphi$, by soundness, there is no model for φ (in which Σ is true). Now note that $\nvdash_\Sigma \neg\varphi$ iff φ is Σ-consistent. (Recall that $\Sigma \nvdash_\Sigma \neg\varphi$ and $\varphi \nvdash_\Sigma \bot$ are equivalent.) So our task is to construct a model for every Σ-consistent formula.

The trick we'll use is to find a Σ-consistent set of formulas that contains φ, but also other formulas which tell us what the world that makes φ true has to look like. Such sets are *complete* Σ-consistent sets. It's not enough to construct a model with a single world to make φ true, it will have to contain multiple worlds and an accessibility relation. The complete Σ-consistent set containing φ will also contain other formulas of the form $\Box\psi$ and $\Diamond\chi$. In all accessible worlds, ψ has to be true; in at least one, χ has to be true. In order to accomplish this, we'll simply take *all* possible complete Σ-consistent sets as the basis for the set of worlds. A tricky part will be to figure out when a complete Σ-consistent set should count as being accessible from another in our model.

We'll show that in the model so defined, φ is true at a world—which is also a complete Σ-consistent set—iff φ is an element of that set. If φ is Σ-consistent, it will be an element of at least one complete Σ-consistent set (a fact we'll prove), and so there will be a world where φ is true. So we will have a single model where every Σ-consistent formula φ is true at some world. This single model is the *canonical* model for Σ.

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Bibliography