com.1 Frame Completeness

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The completeness theorem for **K** can be extended to other modal systems, once we show that the canonical model for a given logic has the corresponding frame property.

nml:com:fra:thm:completeframeprops

Theorem com.1. *If a normal modal logic Σ contains one of the formulas on the left-hand side of Table 1, then the canonical model for Σ has the corresponding property on the right-hand side.*

If Σ contains ...	*... the canonical model for Σ is:*
D: $\Box\varphi \to \Diamond\varphi$	serial;
T: $\Box\varphi \to \varphi$	reflexive;
B: $\varphi \to \Box\Diamond\varphi$	symmetric;
4: $\Box\varphi \to \Box\Box\varphi$	transitive;
5: $\Diamond\varphi \to \Box\Diamond\varphi$	euclidean.

Table 1: Basic correspondence facts.

nml:com:fra:tab:correspondencetable

Proof. We take each of these up in turn.

Suppose Σ contains D, and let $\Delta \in W^\Sigma$; we need to show that there is a Δ' such that $R^\Sigma\Delta\Delta'$. It suffices to show that $\Box^{-1}\Delta$ is Σ-consistent, for then by Lindenbaum's Lemma, there is a complete Σ-consistent set $\Delta' \supseteq \Box^{-1}\Delta$, and by definition of R^Σ we have $R^\Sigma\Delta\Delta'$. So, suppose for contradiction that $\Box^{-1}\Delta$ is *not* Σ-consistent, i.e., $\Box^{-1}\Delta \vdash_\Sigma \bot$. By **??**, $\Delta \vdash_\Sigma \Box\bot$, and since Σ contains D, also $\Delta \vdash_\Sigma \Diamond\bot$. But Σ is normal, so $\Sigma \vdash \neg\Diamond\bot$ (**??**), whence also $\Delta \vdash_\Sigma \neg\Diamond\bot$, against the consistency of Δ.

Now suppose Σ contains T, and let $\Delta \in W^\Sigma$. We want to show $R^\Sigma\Delta\Delta$, i.e., $\Box^{-1}\Delta \subseteq \Delta$. But if $\Box\varphi \in \Delta$ then by T also $\varphi \in \Delta$, as desired.

Now suppose Σ contains B, and suppose $R^\Sigma\Delta\Delta'$ for Δ, $\Delta' \in W^\Sigma$. We need to show that $R^\Sigma\Delta'\Delta$, i.e., $\Box^{-1}\Delta' \subseteq \Delta$. By **??**, this is equivalent to $\Diamond\Delta \subseteq \Delta'$. So suppose $\varphi \in \Delta$. By B, also $\Box\Diamond\varphi \in \Delta$. By the hypothesis that $R^\Sigma\Delta\Delta'$, we have that $\Box^{-1}\Delta \subseteq \Delta'$, and hence $\Diamond\varphi \in \Delta'$, as required.

Now suppose Σ contains 4, and suppose $R^\Sigma\Delta_1\Delta_2$ and $R^\Sigma\Delta_2\Delta_3$. We need to show $R^\Sigma\Delta_1\Delta_3$. From the hypothesis we have both $\Box^{-1}\Delta_1 \subseteq \Delta_2$ and $\Box^{-1}\Delta_2 \subseteq \Delta_3$. In order to show $R^\Sigma\Delta_1\Delta_3$ it suffices to show $\Box^{-1}\Delta_1 \subseteq \Delta_3$. So let $\psi \in \Box^{-1}\Delta_1$, i.e., $\Box\psi \in \Delta_1$. By 4, also $\Box\Box\psi \in \Delta_1$ and by hypothesis we get, first, that $\Box\psi \in \Delta_2$ and, second, that $\psi \in \Delta_3$, as desired.

Now suppose Σ contains 5, suppose $R^\Sigma\Delta_1\Delta_2$ and $R^\Sigma\Delta_1\Delta_3$. We need to show $R^\Sigma\Delta_2\Delta_3$. The first hypothesis gives $\Box^{-1}\Delta_1 \subseteq \Delta_2$, and the second hypothesis is equivalent to $\Diamond\Delta_3 \subseteq \Delta_2$, by **??**. To show $R^\Sigma\Delta_2\Delta_3$, by **??**, it suffices to show $\Diamond\Delta_3 \subseteq \Delta_2$. So let $\Diamond\varphi \in \Diamond\Delta_3$, i.e., $\varphi \in \Delta_3$. By the second hypothesis $\Diamond\varphi \in \Delta_1$ and by 5, $\Box\Diamond\varphi \in \Delta_1$ as well. But now the first hypothesis gives $\Diamond\varphi \in \Delta_2$, as desired. □

As a corollary we obtain completeness results for a number of systems. For instance, we know that **S5** = **KT5** = **KTB4** is complete with respect to the class of all reflexive euclidean models, which is the same as the class of all reflexive, symmetric and transitive models.

Theorem com.2. *Let $\mathcal{C}_{\mathrm{D}}$, $\mathcal{C}_{\mathrm{T}}$, $\mathcal{C}_{\mathrm{B}}$, $\mathcal{C}_4$, and $\mathcal{C}_5$ be the class of all serial, reflexive, symmetric, transitive, and euclidean models (respectively). Then for any schemas $\varphi_1, \dots, \varphi_n$ among* D, T, B, 4, *and* 5, *the system $\mathbf{K}\varphi_1 \dots \varphi_n$ is determined by the class of models $\mathcal{C} = \mathcal{C}_{\varphi_1} \cap \dots \cap \mathcal{C}_{\varphi_n}$.* nml:com:fra:thm:generaldet

Proposition com.3. *Let Σ be a normal modal logic; then:*

1. *If Σ contains the schema $\Diamond\varphi \to \Box\varphi$ then the canonical model for Σ is partially functional.* nml:com:fra:prop:anotherfive-a

2. *If Σ contains the schema $\Diamond\varphi \leftrightarrow \Box\varphi$ then the canonical model for Σ is functional.*

3. *If Σ contains the schema $\Box\Box\varphi \to \Box\varphi$ then the canonical model for Σ is weakly dense.*

(see **??** *for definitions of these frame properties).*

Proof. 1. Suppose that Σ contains the schema $\Diamond\varphi \to \Box\varphi$, to show that R^Σ is partially functional we need to prove that for any Δ_1, Δ_2, $\Delta_3 \in W^\Sigma$, if $R^\Sigma\Delta_1\Delta_2$ and $R^\Sigma\Delta_1\Delta_3$ then $\Delta_2 = \Delta_3$. Since $R^\Sigma\Delta_1\Delta_2$ we have $\Box^{-1}\Delta_1 \subseteq \Delta_2$ and since $R^\Sigma\Delta_1\Delta_3$ also $\Box^{-1}\Delta_1 \subseteq \Delta_3$. The identity $\Delta_2 = \Delta_3$ will follow if we can establish the two inclusions $\Delta_2 \subseteq \Delta_3$ and $\Delta_3 \subseteq \Delta_2$. For the first inclusion, let $\varphi \in \Delta_2$; then $\Diamond\varphi \in \Delta_1$, and by the schema and deductive closure of Δ_1 also $\Box\varphi \in \Delta_1$, whence by the hypothesis that $R^\Sigma\Delta_1\Delta_3$, $\varphi \in \Delta_3$. The second inclusion is similar.

2. This follows immediately from part (1) and the seriality proof in Theorem com.1.

3. Suppose Σ contains the schema $\Box\Box\varphi \to \Box\varphi$ and to show that R^Σ is weakly dense, let $R^\Sigma\Delta_1\Delta_2$. We need to show that there is a complete Σ-consistent set Δ_3 such that $R^\Sigma\Delta_1\Delta_3$ and $R^\Sigma\Delta_3\Delta_2$. Let:

$$\Gamma = \Box^{-1}\Delta_1 \cup \Diamond\Delta_2.$$

It suffices to show that Γ is Σ-consistent, for then by Lindenbaum's Lemma it can be extended to a complete Σ-consistent set Δ_3 such that $\Box^{-1}\Delta_1 \subseteq \Delta_3$ and $\Diamond\Delta_2 \subseteq \Delta_3$, i.e., $R^\Sigma\Delta_1\Delta_3$ and $R^\Sigma\Delta_3\Delta_2$ (by **??**).

Suppose for contradiction that Γ is not consistent. Then there are formulas $\Box\varphi_1, \dots, \Box\varphi_n \in \Delta_1$ and $\psi_1, \dots, \psi_m \in \Delta_2$ such that

$$\varphi_1, \dots, \varphi_n, \Diamond\psi_1, \dots, \Diamond\psi_m \vdash_\Sigma \bot.$$

Since $\Diamond(\psi_1 \land \dots \land \psi_m) \to (\Diamond\psi_1 \land \dots \land \Diamond\psi_m)$ is derivable in every normal modal logic, we argue as follows, contradicting the consistency of Δ_2:

$$
\begin{aligned}
\varphi_1, \dots, \varphi_n, \Diamond\psi_1, \dots, \Diamond\psi_m &\vdash_\Sigma \bot \\
\varphi_1, \dots, \varphi_n &\vdash_\Sigma (\Diamond\psi_1 \land \dots \land \Diamond\psi_m) \to \bot \\
&\quad \text{by the deduction theorem} \\
&\quad \textbf{????}\text{, and TAUT} \\
\varphi_1, \dots, \varphi_n &\vdash_\Sigma \Diamond(\psi_1 \land \dots \land \psi_m) \to \bot \\
&\quad \text{since } \Sigma \text{ is normal} \\
\varphi_1, \dots, \varphi_n &\vdash_\Sigma \neg\Diamond(\psi_1 \land \dots \land \psi_m) \\
&\quad \text{by PL} \\
\varphi_1, \dots, \varphi_n &\vdash_\Sigma \Box\neg(\psi_1 \land \dots \land \psi_m) \\
&\quad \Box\neg \text{ for } \neg\Diamond \\
\Box\varphi_1, \dots, \Box\varphi_n &\vdash_\Sigma \Box\Box\neg(\psi_1 \land \dots \land \psi_m) \\
&\quad \text{by } \textbf{??} \\
\Box\varphi_1, \dots, \Box\varphi_n &\vdash_\Sigma \Box\neg(\psi_1 \land \dots \land \psi_m) \\
&\quad \text{by schema } \Box\Box\varphi \to \Box\varphi \\
\Delta_1 &\vdash_\Sigma \Box\neg(\psi_1 \land \dots \land \psi_m) \\
&\quad \text{by monotonicity, } \textbf{????} \\
\Box\neg(\psi_1 \land \dots \land \psi_m) &\in \Delta_1 \\
&\quad \text{by deductive closure;} \\
\neg(\psi_1 \land \dots \land \psi_m) &\in \Delta_2 \\
&\quad \text{since } R^\Sigma\Delta_1\Delta_2.
\end{aligned}
$$

□

On the strength of these examples, one might think that every system Σ of modal logic is *complete*, in the sense that it proves every formula which is valid in every frame in which every theorem of Σ is valid. Unfortunately, there are many systems that are not complete in this sense.

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Bibliography