com.1 Frame Completeness

The completeness theorem for $\mathbf{K}$ can be extended to other modal systems, once we show that the canonical model for a given logic has the corresponding frame property.

**Theorem com.1.** If a normal modal logic $\Sigma$ contains one of the formulas on the left-hand side of Table 1, then the canonical model for $\Sigma$ has the corresponding property on the right-hand side.

<table>
<thead>
<tr>
<th>If $\Sigma$ contains ...</th>
<th>... the canonical model for $\Sigma$ is:</th>
</tr>
</thead>
<tbody>
<tr>
<td>D: $\Box\varphi \rightarrow \Diamond\varphi$</td>
<td>serial;</td>
</tr>
<tr>
<td>T: $\Box\varphi \rightarrow \varphi$</td>
<td>reflexive;</td>
</tr>
<tr>
<td>B: $\varphi \rightarrow \Box\Diamond\varphi$</td>
<td>symmetric;</td>
</tr>
<tr>
<td>4: $\Box\varphi \rightarrow \Box\Box\varphi$</td>
<td>transitive;</td>
</tr>
<tr>
<td>5: $\Diamond\varphi \rightarrow \Box\Diamond\varphi$</td>
<td>euclidean.</td>
</tr>
</tbody>
</table>

Table 1: Basic correspondence facts.

**Proof.** We take each of these up in turn.

Suppose $\Sigma$ contains D, and let $\Delta \in W^\Sigma$; we need to show that there is a $\Delta'$ such that $R^\Sigma\Delta\Delta'$. It suffices to show that $\Box^{-1}\Delta$ is $\Sigma$-consistent, for then by Lindenbaum’s Lemma, there is a complete $\Sigma$-consistent set $\Delta' \supseteq \Box^{-1}\Delta$, and by definition of $R^\Sigma$ we have $R^\Sigma\Delta\Delta'$. So, suppose for contradiction that $\Box^{-1}\Delta$ is not $\Sigma$-consistent, i.e., $\Box^{-1}\Delta \vdash_{\Sigma} \bot$. By $??$, $\Delta \vdash_{\Sigma} \Box\bot$, and since $\Sigma$ contains D, also $\Delta \vdash_{\Sigma} \Diamond\bot$. But $\Sigma$ is normal, so $\Sigma \vdash_{\Sigma} \neg \Diamond\bot$ ($??$), whence also $\Delta \vdash_{\Sigma} \neg \Diamond\bot$, against the consistency of $\Delta$.

Now suppose $\Sigma$ contains T, and let $\Delta \in W^\Sigma$. We want to show $R^\Sigma\Delta\Delta$, i.e., $\Box^{-1}\Delta \subseteq \Delta$. But if $\Box\varphi \in \Delta$ then by T also $\varphi \in \Delta$, as desired.

Now suppose $\Sigma$ contains B, and suppose $R^\Sigma\Delta\Delta'$ for $\Delta, \Delta' \in W^\Sigma$. We need to show that $R^\Sigma\Delta'\Delta$, i.e., $\Box^{-1}\Delta' \subseteq \Delta$. By $??$, this is equivalent to $\Diamond\Delta \subseteq \Delta'$. So suppose $\varphi \in \Delta$. By B, also $\Box\Diamond\varphi \in \Delta$. By the hypothesis that $R^\Sigma\Delta\Delta'$, we have that $\Box^{-1}\Delta \subseteq \Delta'$, and hence $\Diamond\varphi \in \Delta'$, as required.

Now suppose $\Sigma$ contains 4, and suppose $R^\Sigma\Delta_1\Delta_2$ and $R^\Sigma\Delta_2\Delta_3$. We need to show $R^\Sigma\Delta_1\Delta_3$. From the hypothesis we have both $\Box^{-1}\Delta_1 \subseteq \Delta_2$ and $\Box^{-1}\Delta_2 \subseteq \Delta_3$. In order to show $R^\Sigma\Delta_1\Delta_3$ it suffices to show $\Box^{-1}\Delta_1 \subseteq \Delta_3$. So let $\psi \in \Box^{-1}\Delta_1$, i.e., $\Box\psi \in \Delta_1$. By 4, also $\Box\Box\psi \in \Delta_1$ and by hypothesis we get, first, that $\Box\psi \in \Delta_2$ and, second, that $\psi \in \Delta_3$, as desired.

Now suppose $\Sigma$ contains 5, suppose $R^\Sigma\Delta_1\Delta_2$ and $R^\Sigma\Delta_1\Delta_3$. We need to show $R^\Sigma\Delta_2\Delta_3$. The first hypothesis gives $\Box^{-1}\Delta_1 \subseteq \Delta_2$, and the second hypothesis is equivalent to $\Diamond\Delta_3 \subseteq \Delta_2$, by $??$. To show $R^\Sigma\Delta_2\Delta_3$, by $??$, it suffices to show $\Diamond\Delta_3 \subseteq \Delta_2$. So let $\Diamond\varphi \in \Diamond\Delta_3$, i.e., $\varphi \in \Delta_3$. By the second hypothesis $\Diamond\varphi \in \Delta_1$ and by 5, $\Box\Diamond\varphi \in \Delta_1$ as well. But now the first hypothesis gives $\Diamond\varphi \in \Delta_2$, as desired. □

As a corollary we obtain completeness results for a number of systems. For instance, we know that $\mathbf{S}5 = \mathbf{KT}5 = \mathbf{KT}4$ is complete with respect to the
class of all reflexive euclidean models, which is the same as the class of all reflexive, symmetric and transitive models.

**Theorem com.2.** Let $C_D$, $C_T$, $C_B$, $C_4$, and $C_5$ be the class of all serial, reflexive, symmetric, transitive, and euclidean models (respectively). Then for any schemas $\varphi_1$, \ldots, $\varphi_n$ among $D$, $T$, $B$, $4$, and $5$, the system $K\varphi_1 \ldots \varphi_n$ is determined by the class of models $C = C_{\varphi_1} \cap \cdots \cap C_{\varphi_n}$.

**Proposition com.3.** Let $\Sigma$ be a normal modal logic; then:

1. If $\Sigma$ contains the schema $\Box \varphi \rightarrow \Box \varphi$ then the canonical model for $\Sigma$ is partially functional.

2. If $\Sigma$ contains the schema $\Box \varphi \leftrightarrow \Box \varphi$ then the canonical model for $\Sigma$ is functional.

3. If $\Sigma$ contains the schema $\Box \Box \varphi \rightarrow \Box \varphi$ then the canonical model for $\Sigma$ is weakly dense.

(see ?? for definitions of these frame properties).

**Proof.** 1. Suppose that $\Sigma$ contains the schema $\Box \varphi \rightarrow \Box \varphi$, to show that $R^\Sigma$ is partially functional we need to prove that for any $\Delta_1$, $\Delta_2$, $\Delta_3 \in W^\Sigma$, if $R^\Sigma \Delta_1 \Delta_2$ and $R^\Sigma \Delta_1 \Delta_3$ then $\Delta_2 = \Delta_3$. Since $R^\Sigma \Delta_1 \Delta_2$ we have $\Box^{-1} \Delta_1 \subseteq \Delta_2$ and since $R^\Sigma \Delta_1 \Delta_3$ also $\Box^{-1} \Delta_1 \subseteq \Delta_3$. The identity $\Delta_2 = \Delta_3$ will follow if we can establish the two inclusions $\Delta_2 \subseteq \Delta_3$ and $\Delta_3 \subseteq \Delta_2$. For the first inclusion, let $\varphi \in \Delta_2$; then $\Box \varphi \in \Delta_1$, and by the schema and deductive closure of $\Delta_1$ also $\Box \varphi \in \Delta_1$, whence by the hypothesis that $R^\Sigma \Delta_1 \Delta_3$, $\varphi \in \Delta_3$. The second inclusion is similar.

2. This follows immediately from part (1) and the seriality proof in Theorem com.1.

3. Suppose $\Sigma$ contains the schema $\Box \Box \varphi \rightarrow \Box \varphi$ and to show that $R^\Sigma$ is weakly dense, let $R^\Sigma \Delta_1 \Delta_2$. We need to show that there is a complete $\Sigma$-consistent set $\Delta_3$ such that $R^\Sigma \Delta_1 \Delta_3$ and $R^\Sigma \Delta_3 \Delta_2$. Let:

$$\Gamma = \Box^{-1} \Delta_1 \cup \Box \Delta_2.$$ 

It suffices to show that $\Gamma$ is $\Sigma$-consistent, for then by Lindenbaum’s Lemma it can be extended to a complete $\Sigma$-consistent set $\Delta_3$ such that $\Box^{-1} \Delta_1 \subseteq \Delta_3$ and $\Box \Delta_2 \subseteq \Delta_3$, i.e., $R^\Sigma \Delta_1 \Delta_3$ and $R^\Sigma \Delta_3 \Delta_2$ (by ??).

Suppose for contradiction that $\Gamma$ is not consistent. Then there are formulas $\Box \varphi_1$, \ldots, $\Box \varphi_n \in \Delta_1$ and $\psi_1$, \ldots, $\psi_m \in \Delta_2$ such that

$$\varphi_1, \ldots, \varphi_n, \Box \psi_1, \ldots, \Box \psi_m \vdash \perp.$$
Since $◊(ψ_1 ∧ ⋯ ∧ ψ_m) → (◊ψ_1 ∧ ⋯ ∧ ◊ψ_m)$ is derivable in every normal modal logic, we argue as follows, contradicting the consistency of $Δ_2$:

\[
\begin{align*}
φ_1, \ldots, φ_n, ◊ψ_1, \ldots, ◊ψ_m & \vdash ⊥ \\
φ_1, \ldots, φ_n & \vdash_Σ (◊ψ_1 ∧ ⋯ ∧ ◊ψ_m) → ⊥ \\
& \text{by the deduction theorem} \\
& \text{TAUT}
\end{align*}
\]

\[
\begin{align*}
φ_1, \ldots, φ_n & \vdash_Σ ◊(ψ_1 ∧ ⋯ ∧ ψ_m) → ⊥ \\
& \text{since $Σ$ is normal}
\end{align*}
\]

\[
φ_1, \ldots, φ_n & \vdash_Σ ¬◊(ψ_1 ∧ ⋯ ∧ ψ_m) \\
& \text{by PL}
\]

\[
\begin{align*}
□φ_1, \ldots, □φ_n & \vdash_Σ □¬(ψ_1 ∧ ⋯ ∧ ψ_m) \\
& \text{by ??}
\end{align*}
\]

\[
\begin{align*}
□φ_1, \ldots, □φ_n & \vdash_Σ □¬(ψ_1 ∧ ⋯ ∧ ψ_m) \\
& \text{by schema □□φ → □φ}
\end{align*}
\]

\[
\begin{align*}
Δ_1 & \vdash_Σ □¬(ψ_1 ∧ ⋯ ∧ ψ_m) \\
& \text{by monotonicity, ???}
\end{align*}
\]

\[
\begin{align*}
□¬(ψ_1 ∧ ⋯ ∧ ψ_m) & \in Δ_1 \\
& \text{by deductive closure;}
\end{align*}
\]

\[
\begin{align*}
¬(ψ_1 ∧ ⋯ ∧ ψ_m) & \in Δ_2 \\
& \text{since $R^Σ Δ_1 Δ_2$.}
\end{align*}
\]

On the strength of these examples, one might think that every system $Σ$ of modal logic is complete, in the sense that it proves every formula which is valid in every frame in which every theorem of $Σ$ is valid. Unfortunately, there are many systems that are not complete in this sense.

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**Bibliography**