Suppose $\Sigma$ is a set of modal formulas—think of them as the axioms or defining principles of a normal modal logic. A set $\Gamma$ is $\Sigma$-consistent iff $\Gamma \not\vdash \bot$, i.e., if there is no derivation of $\varphi_1 \to (\varphi_2 \to (\cdots (\varphi_n \to \bot) \cdots)$ from $\Sigma$, where each $\varphi_i \in \Gamma$. We will construct a “canonical” model in which each world is taken to be a special kind of $\Sigma$-consistent set: one which is not just $\Sigma$-consistent, but maximally so, in the sense that it settles the truth value of every modal formula: for every $\varphi$, either $\varphi \in \Gamma$ or $\neg \varphi \in \Gamma$:

**Definition com.1.** A set $\Gamma$ is complete $\Sigma$-consistent if and only if it is $\Sigma$-consistent and for every $\varphi$, either $\varphi \in \Gamma$ or $\neg \varphi \in \Gamma$.

Complete $\Sigma$-consistent sets $\Gamma$ have a number of useful properties. For one, they are deductively closed, i.e., if $\Gamma \vdash \Sigma \varphi$ then $\varphi \in \Gamma$. This means in particular that every instance of a formula $\varphi \in \Sigma$ is also $\in \Gamma$. Moreover, membership in $\Gamma$ mirrors the truth conditions for the propositional connectives. This will be important when we define the “canonical model.”

**Proposition com.2.** Suppose $\Gamma$ is complete $\Sigma$-consistent. Then:

1. $\Gamma$ is deductively closed in $\Sigma$.
2. $\Sigma \subseteq \Gamma$.
3. $\bot \notin \Gamma$.
4. $\top \in \Gamma$.
5. $\neg \varphi \in \Gamma$ if and only if $\varphi \notin \Gamma$.
6. $\varphi \land \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$.
7. $\varphi \lor \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$.
8. $\varphi \to \psi \in \Gamma$ iff $\varphi \notin \Gamma$ or $\psi \in \Gamma$.
9. $\varphi \leftrightarrow \psi \in \Gamma$ iff either $\varphi \in \Gamma$ and $\psi \in \Gamma$, or $\varphi \notin \Gamma$ and $\psi \notin \Gamma$.

**Proof.**

1. Suppose $\Gamma \vdash _\Sigma \varphi$ but $\varphi \notin \Gamma$. Then since $\Gamma$ is complete $\Sigma$-consistent, $\neg \varphi \in \Gamma$. This would make $\Gamma$ inconsistent, since $\varphi, \neg \varphi \vdash _\Sigma \bot$.
2. If $\varphi \in \Sigma$ then $\Gamma \vdash _\Sigma \varphi$, and $\varphi \in \Gamma$ by deductive closure, i.e., case (1).
3. If $\bot \in \Gamma$, then $\Gamma \vdash _\Sigma \bot$, so $\Gamma$ would be $\Sigma$-inconsistent.
4. $\Gamma \vdash _\Sigma \top$, so $\top \in \Gamma$ by deductive closure, i.e., case (1).
5. If $\neg \varphi \in \Gamma$, then by consistency $\varphi \notin \Gamma$; and if $\varphi \notin \Gamma$ then $\varphi \in \Gamma$ since $\Gamma$ is complete $\Sigma$-consistent.
6. Suppose $\varphi \land \psi \in \Gamma$. Since $(\varphi \land \psi) \rightarrow \varphi$ is a tautological instance, $\varphi \in \Gamma$ by deductive closure, i.e., case (1). Similarly for $\psi \in \Gamma$. On the other hand, suppose both $\varphi \in \Gamma$ and $\psi \in \Gamma$. Then deductive closure implies $(\varphi \land \psi) \in \Gamma$, since $\varphi \rightarrow (\psi \rightarrow (\varphi \land \psi))$ is a tautological instance.

7. Suppose $\varphi \lor \psi \in \Gamma$, and $\varphi \notin \Gamma$ and $\psi \notin \Gamma$. Since $\Gamma$ is complete $\Sigma$-consistent, $\neg \varphi \in \Gamma$ and $\neg \psi \in \Gamma$. Then $\neg(\varphi \lor \psi) \in \Gamma$ since $\neg \varphi \rightarrow (\neg \psi \rightarrow \neg(\varphi \lor \psi))$ is a tautological instance. This would mean that $\Gamma$ is $\Sigma$-inconsistent, a contradiction.

8. Suppose $\varphi \rightarrow \psi \in \Gamma$ and $\varphi \notin \Gamma$; then $\Gamma \vdash \Sigma \psi$, whence $\psi \in \Gamma$ by deductive closure. Conversely, if $\varphi \rightarrow \psi \notin \Gamma$ then since $\Gamma$ is complete $\Sigma$-consistent, $\neg(\varphi \rightarrow \psi) \in \Gamma$. Since $\neg(\varphi \rightarrow \psi) \rightarrow \varphi$ is a tautological instance, $\varphi \in \Gamma$ by deductive closure. Since $\neg(\varphi \rightarrow \psi) \rightarrow \neg \psi$ is a tautological instance, $\neg \psi \in \Gamma$. Then $\psi \notin \Gamma$ since $\Gamma$ is $\Sigma$-consistent.

9. Suppose $\varphi \leftrightarrow \psi \in \Gamma$. If $\varphi \in \Gamma$, then $\psi \in \Gamma$, since $(\varphi \leftrightarrow \psi) \rightarrow (\varphi \rightarrow \psi)$ is a tautological instance. Similarly, if $\psi \in \Gamma$, then $\varphi \in \Gamma$. So either both $\varphi \in \Gamma$ and $\psi \in \Gamma$, or neither $\varphi \in \Gamma$ nor $\psi \in \Gamma$.

Conversely, suppose $\varphi \rightarrow \psi \notin \Gamma$. Since $\Gamma$ is complete $\Sigma$-consistent, $\neg(\varphi \leftrightarrow \psi) \in \Gamma$. Since $\neg(\varphi \leftrightarrow \psi) \rightarrow (\varphi \rightarrow \neg \psi)$ is a tautological instance, if $\varphi \in \Gamma$ then $\neg \psi \in \Gamma$, and since $\Gamma$ is $\Sigma$-consistent, $\psi \notin \Gamma$. Similarly, if $\psi \in \Gamma$ then $\varphi \notin \Gamma$. So neither $\varphi \in \Gamma$ and $\psi \in \Gamma$, nor $\varphi \notin \Gamma$ and $\psi \notin \Gamma$. \hfill \Box

**Problem com.1.** Complete the proof of Proposition com.2.

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