prf.1 Showing Systems are Distinct

nml:prf:dis: sec In **??** we saw how to prove that two systems of modal logic are in fact the same system. **??** allows us to show that two modal systems Σ and Σ' are distinct, by finding a formula φ such that $\Sigma' \vdash \varphi$ that fails in a model of Σ.

Proposition prf.1. $\mathbf{KD} \subsetneq \mathbf{KT}$

Proof. This is the syntactic counterpart to the semantic fact that all reflexive relations are serial. To show $\mathbf{KD} \subseteq \mathbf{KT}$ we need to see that $\mathbf{KD} \vdash \psi$ implies $\mathbf{KT} \vdash \psi$, which follows from $\mathbf{KT} \vdash \mathbf{D}$, as shown in **????**. To show that the inclusion is proper, by Soundness (**??**), it suffices to exhibit a model of **KD** where T, i.e., $\Box p \to p$, fails (an easy task left as an exercise), for then by Soundness $\mathbf{KD} \nvdash \Box p \to p$. □

Proposition prf.2. $\mathbf{KB} \neq \mathbf{K4}$.

Proof. We construct a symmetric model where some instance of 4 fails; since obviously the instance is derivable for **K4** but not in **KB**, it will follow $\mathbf{K4} \nsubseteq \mathbf{KB}$. Consider the symmetric model $\mathfrak{M}$ of Figure 1. Since the model is symmetric, K and B are true in $\mathfrak{M}$ (by **??** and **??**, respectively). However, $\mathfrak{M}, w_1 \nVdash \Box p \to \Box\Box p$. □

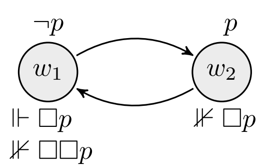

Figure 1: A symmetric model falsifying an instance of 4.

nml:prf:dis: fig:Bnot4 nml:prf:dis: thm:KTBnot45 **Theorem prf.3.** $\mathbf{KTB} \nvdash \mathbf{4}$ *and* $\mathbf{KTB} \nvdash \mathbf{5}$.

Proof. By **??** we know that all instances of T and B are true in every reflexive symmetric model (respectively). So by soundness, it suffices to find a reflexive symmetric model containing a world at which some instance of 4 fails, and similarly for 5. We use the same model for both claims. Consider the symmetric, reflexive model in Figure 2. Then $\mathfrak{M}, w_1 \nVdash \Box p \to \Box\Box p$, so 4 fails at w_1. Similarly, $\mathfrak{M}, w_2 \nVdash \Diamond\neg p \to \Box\Diamond\neg p$, so the instance of 5 with $\varphi = \neg p$ fails at w_2. □

nml:prf:dis: thm:KD5not4 **Theorem prf.4.** $\mathbf{KD5} \neq \mathbf{KT4} = \mathbf{S4}$.

Proof. By **??** we know that all instances of D and 5 are true in all serial euclidean models. So it suffices to find a serial euclidean model containing a world at which some instance of 4 fails. Consider the model of Figure 3, and notice that $\mathfrak{M}, w_1 \nVdash \Box p \to \Box\Box p$. □

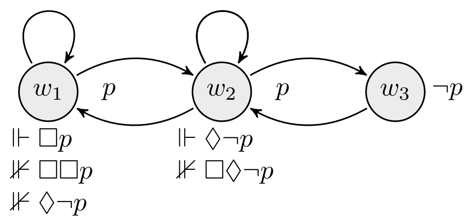

Figure 2: The model for Theorem prf.3.

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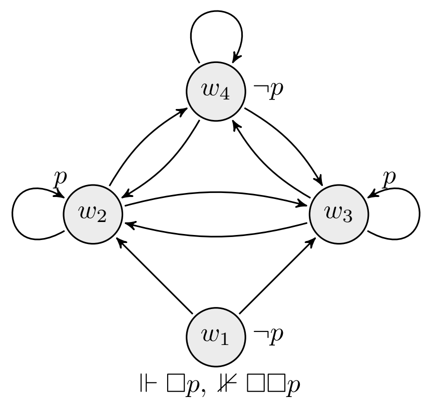

Figure 3: The model for Theorem prf.4.

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Problem prf.1. Give an alternative proof of Theorem prf.4 using a model with 3 worlds.

Problem prf.2. Provide a single reflexive transitive model showing that both **KT4** $\nvdash$ B and **KT4** $\nvdash$ 5.

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Bibliography