

prf.1 Showing Systems are Distinct

In ?? we saw how to prove that two systems of modal logic are in fact the same system. ?? allows us to show that two modal systems Σ and Σ' are distinct, by finding a formula φ such that $\Sigma' \vdash \varphi$ that fails in a model of Σ .

Proposition prf.1. $\mathbf{KD} \subsetneq \mathbf{KT}$

Proof. This is the syntactic counterpart to the semantic fact that all reflexive relations are serial. To show $\mathbf{KD} \subseteq \mathbf{KT}$ we need to see that $\mathbf{KD} \vdash \psi$ implies $\mathbf{KT} \vdash \psi$, which follows from $\mathbf{KT} \vdash \mathbf{D}$, as shown in ????. To show that the inclusion is proper, by Soundness (??), it suffices to exhibit a model of \mathbf{KD} where \mathbf{T} , i.e., $\Box p \rightarrow p$, fails (an easy task left as an exercise), for then by Soundness $\mathbf{KD} \not\vdash \Box p \rightarrow p$. \square

Proposition prf.2. $\mathbf{KB} \neq \mathbf{K4}$.

Proof. We construct a symmetric model where some instance of 4 fails; since obviously the instance is derivable for $\mathbf{K4}$ but not in \mathbf{KB} , it will follow $\mathbf{K4} \not\subseteq \mathbf{KB}$. Consider the symmetric model \mathfrak{M} of Figure 1. Since the model is symmetric, \mathbf{K} and \mathbf{B} are true in \mathfrak{M} (by ?? and ??, respectively). However, $\mathfrak{M}, w_1 \not\models \Box p \rightarrow \Box \Box p$. \square

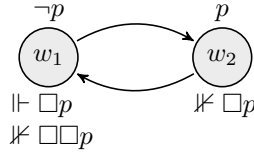


Figure 1: A symmetric model falsifying an instance of 4.

nml:prf:dis:
nml:prf:dis:
fig:Bnot4
thm:KTBnot45

Theorem prf.3. $\mathbf{KTB} \not\vdash 4$ and $\mathbf{KTB} \not\vdash 5$.

Proof. By ?? we know that all instances of \mathbf{T} and \mathbf{B} are true in every reflexive symmetric model (respectively). So by soundness, it suffices to find a reflexive symmetric model containing a world at which some instance of 4 fails, and similarly for 5. We use the same model for both claims. Consider the symmetric, reflexive model in Figure 2. Then $\mathfrak{M}, w_1 \not\models \Box p \rightarrow \Box \Box p$, so 4 fails at w_1 . Similarly, $\mathfrak{M}, w_2 \not\models \Diamond \neg p \rightarrow \Box \Diamond \neg p$, so the instance of 5 with $\varphi = \neg p$ fails at w_2 . \square

nml:prf:dis:
thm:KD5not4

Theorem prf.4. $\mathbf{KD5} \neq \mathbf{KT4} = \mathbf{S4}$.

Proof. By ?? we know that all instances of \mathbf{D} and 5 are true in all serial euclidean models. So it suffices to find a serial euclidean model containing a world at which some instance of 4 fails. Consider the model of Figure 3, and notice that $\mathfrak{M}, w_1 \not\models \Box p \rightarrow \Box \Box p$. \square

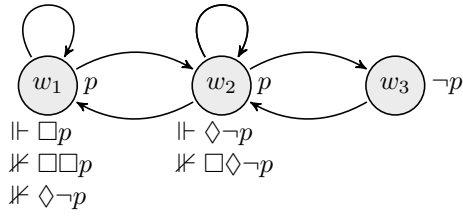


Figure 2: The model for **Theorem prf.3**.

nml:prf:dis:
fig:KT5not45

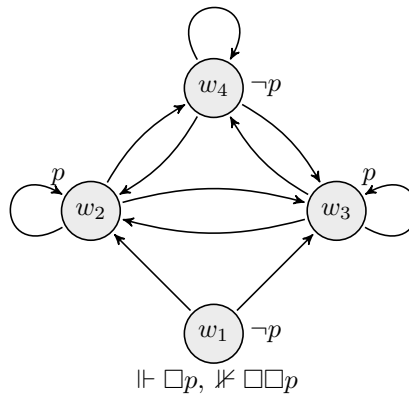


Figure 3: The model for **Theorem prf.4**.

nml:prf:dis:
fig:KD5not4

Problem prf.1. Give an alternative proof of **Theorem prf.4** using a model with 3 worlds.

Problem prf.2. Provide a single reflexive transitive model showing that both **KT4** $\not\vdash$ B and **KT4** $\not\vdash$ 5.

Photo Credits

Bibliography