

## prf.1 Proofs in Modal Systems

nml:prf:prs:sec We now come to proofs in systems of modal logic other than **K**.

nml:prf:prs:prop:S5facts **Proposition prf.1.** *The following provability results obtain:*

1. **KT5**  $\vdash$  B;
2. **KT5**  $\vdash$  4;
3. **KDB4**  $\vdash$  T;
4. **KB4**  $\vdash$  5;
5. **KB5**  $\vdash$  4;
6. **KT**  $\vdash$  D.

nml:prf:prs:prop:S5facts-KT-D

*Proof.* We exhibit proofs for each.

1. **KT5**  $\vdash$  B:

1. **KT5**  $\vdash$   $\diamond\varphi \rightarrow \Box\diamond\varphi$  5
2. **KT5**  $\vdash$   $\varphi \rightarrow \diamond\varphi$   $T_\diamond$
3. **KT5**  $\vdash$   $\varphi \rightarrow \Box\diamond\varphi$  PL.

2. **KT5**  $\vdash$  4:

1. **KT5**  $\vdash$   $\diamond\Box\varphi \rightarrow \Box\diamond\Box\varphi$  5 with  $\Box\varphi$  for  $p$
2. **KT5**  $\vdash$   $\Box\varphi \rightarrow \diamond\Box\varphi$   $T_\diamond$  with  $\Box\varphi$  for  $p$
3. **KT5**  $\vdash$   $\Box\varphi \rightarrow \Box\diamond\Box\varphi$  PL, 1, 2
4. **KT5**  $\vdash$   $\diamond\Box\varphi \rightarrow \Box\varphi$   $5_\diamond$
5. **KT5**  $\vdash$   $\Box\diamond\Box\varphi \rightarrow \Box\Box\varphi$  RK, 4
6. **KT5**  $\vdash$   $\Box\varphi \rightarrow \Box\Box\varphi$  PL, 3, 5.

3. **KDB4**  $\vdash$  T:

1. **KDB4**  $\vdash$   $\diamond\Box\varphi \rightarrow \varphi$   $B_\diamond$
2. **KDB4**  $\vdash$   $\Box\Box\varphi \rightarrow \diamond\Box\varphi$  D with  $\Box\varphi$  for  $p$
3. **KDB4**  $\vdash$   $\Box\Box\varphi \rightarrow \varphi$  PL1, 2
4. **KDB4**  $\vdash$   $\Box\varphi \rightarrow \Box\Box\varphi$  4
5. **KDB4**  $\vdash$   $\Box\varphi \rightarrow \varphi$  PL, 1, 4.

4. **KB4**  $\vdash$  5:

1. **KB4**  $\vdash$   $\diamond\varphi \rightarrow \Box\diamond\diamond\varphi$  B with  $\diamond\varphi$  for  $p$
2. **KB4**  $\vdash$   $\diamond\diamond\varphi \rightarrow \diamond\varphi$   $4_\diamond$
3. **KB4**  $\vdash$   $\Box\diamond\diamond\varphi \rightarrow \Box\diamond\varphi$  RK, 2
4. **KB4**  $\vdash$   $\diamond\varphi \rightarrow \Box\diamond\varphi$  PL, 1, 3.

5. **KB5** ⊢ 4:

1. **KB5** ⊢  $\Box\varphi \rightarrow \Box\Diamond\Box\varphi$  B with  $\Box\varphi$  for  $p$
2. **KB5** ⊢  $\Diamond\Box\varphi \rightarrow \Box\varphi$   $5_\Diamond$
3. **KB5** ⊢  $\Box\Diamond\Box\varphi \rightarrow \Box\Box\varphi$  RK, 2
4. **KB5** ⊢  $\Box\varphi \rightarrow \Box\Box\varphi$  PL, 1, 3.

6. **KT** ⊢ D:

1. **KT** ⊢  $\Box\varphi \rightarrow \varphi$  T
2. **KT** ⊢  $\varphi \rightarrow \Diamond\varphi$   $T_\Diamond$
3. **KT** ⊢  $\Box\varphi \rightarrow \Diamond\varphi$  PL, 1, 2 □

**Definition prf.2.** Following tradition, we define **S4** to be the system **KT4**, and **S5** the system **KTB4**.

The following proposition shows that the classical system **S5** has several equivalent axiomatizations. This should not surprise, as the various combinations of axioms all characterize equivalence relations (see ??).

**Proposition prf.3.** **KTB4 = KT5 = KDB4 = KDB5.**

*nml:prf:prs:  
prop:S5*

*Proof.* Exercise. □

**Problem prf.1.** Prove **Proposition prf.3.**

## Photo Credits

## Bibliography