

prf.1 Proofs in K

nml:prf:prk:
sec In order to practice proofs in the smallest modal system, we show the valid formulas on the left-hand side of ?? can all be given K-proofs.

Proposition prf.1. $\mathbf{K} \vdash \Box\varphi \rightarrow \Box(\psi \rightarrow \varphi)$

Proof.

- | | | | |
|----|---|----------|---|
| 1. | $\varphi \rightarrow (\psi \rightarrow \varphi)$ | TAUT | |
| 2. | $\Box(\varphi \rightarrow (\psi \rightarrow \varphi))$ | NEC, 1 | |
| 3. | $\Box(\varphi \rightarrow (\psi \rightarrow \varphi)) \rightarrow (\Box\varphi \rightarrow \Box(\psi \rightarrow \varphi))$ | K | |
| 4. | $\Box\varphi \rightarrow \Box(\psi \rightarrow \varphi)$ | MP, 2, 3 | □ |

Proposition prf.2. $\mathbf{K} \vdash \Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

Proof.

- | | | | |
|-----|--|------------|--|
| 1. | $(\varphi \wedge \psi) \rightarrow \varphi$ | TAUT | |
| 2. | $\Box((\varphi \wedge \psi) \rightarrow \varphi)$ | NEC | |
| 3. | $\Box((\varphi \wedge \psi) \rightarrow \varphi) \rightarrow (\Box(\varphi \wedge \psi) \rightarrow \Box\varphi)$ | K | |
| 4. | $\Box(\varphi \wedge \psi) \rightarrow \Box\varphi$ | MP, 2, 3 | |
| 5. | $(\varphi \wedge \psi) \rightarrow \psi$ | TAUT | |
| 6. | $\Box((\varphi \wedge \psi) \rightarrow \psi)$ | NEC | |
| 7. | $\Box((\varphi \wedge \psi) \rightarrow \psi) \rightarrow (\Box(\varphi \wedge \psi) \rightarrow \Box\psi)$ | K | |
| 8. | $\Box(\varphi \wedge \psi) \rightarrow \Box\psi$ | MP, 6, 7 | |
| 9. | $(\Box(\varphi \wedge \psi) \rightarrow \Box\varphi) \rightarrow$
$((\Box(\varphi \wedge \psi) \rightarrow \Box\psi) \rightarrow$
$(\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)))$ | TAUT | |
| 10. | $(\Box(\varphi \wedge \psi) \rightarrow \Box\psi) \rightarrow$
$(\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi))$ | MP, 4, 9 | |
| 11. | $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$ | MP, 8, 10. | |

Note that the formula on line 9 is an instance of the tautology

$$(p \rightarrow q) \rightarrow ((p \rightarrow r) \rightarrow (p \rightarrow (q \wedge r))). \quad \square$$

Proposition prf.3. $\mathbf{K} \vdash (\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

Proof.

1. $\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$ TAUT
2. $\Box(\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi)))$ NEC, 1
3. $\Box(\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))) \rightarrow (\Box\varphi \rightarrow \Box(\psi \rightarrow (\varphi \wedge \psi)))$ K
4. $\Box\varphi \rightarrow \Box(\psi \rightarrow (\varphi \wedge \psi))$ MP, 2, 3
5. $\Box(\psi \rightarrow (\varphi \wedge \psi)) \rightarrow (\Box\psi \rightarrow \Box(\varphi \wedge \psi))$ K
6. $(\Box\varphi \rightarrow \Box(\psi \rightarrow (\varphi \wedge \psi))) \rightarrow$
 $(\Box(\psi \rightarrow (\varphi \wedge \psi)) \rightarrow (\Box\psi \rightarrow \Box(\varphi \wedge \psi))) \rightarrow$
 $(\Box\varphi \rightarrow (\Box\psi \rightarrow \Box(\varphi \wedge \psi)))$ TAUT
7. $(\Box(\psi \rightarrow (\varphi \wedge \psi)) \rightarrow (\Box\psi \rightarrow \Box(\varphi \wedge \psi))) \rightarrow$
 $(\Box\varphi \rightarrow (\Box\psi \rightarrow \Box(\varphi \wedge \psi)))$ MP, 4, 6
8. $\Box\varphi \rightarrow (\Box\psi \rightarrow \Box(\varphi \wedge \psi))$ MP, 5, 7
9. $(\Box\varphi \rightarrow (\Box\psi \rightarrow \Box(\varphi \wedge \psi))) \rightarrow$
 $((\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi))$ TAUT
10. $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$ MP, 8, 9

The **formulas** on lines 6 and 9 are instances of the tautologies

$$(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$$

$$(p \rightarrow (q \rightarrow r)) \rightarrow ((p \wedge q) \rightarrow r) \quad \square$$

Proposition prf.4. $\mathbf{K} \vdash \neg\Box p \rightarrow \Diamond\neg p$

Proof.

1. $\Diamond\neg p \leftrightarrow \neg\Box\neg\neg p$ DUAL
2. $(\Diamond\neg p \leftrightarrow \neg\Box\neg\neg p) \rightarrow$
 $(\neg\Box\neg\neg p \rightarrow \Diamond\neg p)$ TAUT
3. $\neg\Box\neg\neg p \rightarrow \Diamond\neg p$ MP, 1, 2
4. $\neg\neg p \rightarrow p$ TAUT
5. $\Box(\neg\neg p \rightarrow p)$ NEC, 4
6. $\Box(\neg\neg p \rightarrow p) \rightarrow (\Box\neg\neg p \rightarrow \Box p)$ K
7. $(\Box\neg\neg p \rightarrow \Box p)$ MP, 5, 6
8. $(\Box\neg\neg p \rightarrow \Box p) \rightarrow (\neg\Box p \rightarrow \neg\Box\neg\neg p)$ TAUT
9. $\neg\Box p \rightarrow \neg\Box\neg\neg p$ MP, 7, 8
10. $(\neg\Box p \rightarrow \neg\Box\neg\neg p) \rightarrow$
 $((\neg\Box\neg\neg p \rightarrow \Diamond\neg p) \rightarrow (\neg\Box p \rightarrow \Diamond\neg p))$ TAUT
11. $(\neg\Box\neg\neg p \rightarrow \Diamond\neg p) \rightarrow (\neg\Box p \rightarrow \Diamond\neg p)$ MP, 9, 10
12. $\neg\Box p \rightarrow \Diamond\neg p$ MP, 3, 11

The **formulas** on lines 8 and 10 are instances of the tautologies

$$(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$$

$$(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r)). \quad \square$$

Problem prf.1. Find **derivations** in **K** for the following **formulas**:

1. $\Box\neg p \rightarrow \Box(p \rightarrow q)$

2. $(\Box p \vee \Box q) \rightarrow \Box(p \vee q)$

3. $\Diamond p \rightarrow \Diamond(p \vee q)$

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Bibliography