

prf.1 Normal Modal Logics

nml:prf:nor:
sec

Not every set of modal **formulas** can easily be characterized as those **formulas** derivable from a set of axioms. We want modal logics to be well-behaved. First of all, everything we can **derive** in classical propositional logic should still be **derivable**, of course taking into account that the **formulas** may now contain also \Box and \Diamond . To this end, we require that a modal logic contain all tautological instances and be closed under modus ponens.

Definition prf.1. A *modal logic* is a set Σ of modal **formulas** which

1. contains all tautologies, and
2. is closed under substitution, i.e., if $\varphi \in \Sigma$, and $\theta_1, \dots, \theta_n$ are **formulas**, then

$$\varphi[\theta_1/p_1, \dots, \theta_n/p_n] \in \Sigma,$$

3. is closed under *modus ponens*, i.e., if φ and $\varphi \rightarrow \psi \in \Sigma$, then $\psi \in \Sigma$.

In order to use the relational semantics for modal logics, we also have to require that all **formulas** valid in all modal models are included. It turns out that this requirement is met as soon as all instances of K and DUAL are **derivable**, and whenever a **formula** φ is **derivable**, so is $\Box\varphi$. A modal logic that satisfies these conditions is called *normal*. (Of course, there are also non-normal modal logics, but the usual relational models are not adequate for them.)

Definition prf.2. A modal logic Σ is *normal* if it contains

$$\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q), \quad (\text{K})$$

$$\Diamond p \leftrightarrow \neg \Box \neg p \quad (\text{DUAL})$$

and is closed under *necessitation*, i.e., if $\varphi \in \Sigma$, then $\Box\varphi \in \Sigma$.

Observe that while tautological implication is “fine-grained” enough to preserve *truth at a world*, the rule NEC only preserves *truth in a model* (and hence also validity in a frame or in a class of frames).

nml:prf:nor:
prop:rk

Proposition prf.3. *Every normal modal logic is closed under rule RK,*

$$\frac{\varphi_1 \rightarrow (\varphi_2 \rightarrow \dots (\varphi_{n-1} \rightarrow \varphi_n) \dots)}{\Box\varphi_1 \rightarrow (\Box\varphi_2 \rightarrow \dots (\Box\varphi_{n-1} \rightarrow \Box\varphi_n) \dots)} \text{RK}$$

Proof. By induction on n : If $n = 1$, then the rule is just NEC, and every normal modal logic is closed under NEC.

Now suppose the result holds for $n - 1$; we show it holds for n .

Assume

$$\varphi_1 \rightarrow (\varphi_2 \rightarrow \dots (\varphi_{n-1} \rightarrow \varphi_n) \dots) \in \Sigma$$

By the induction hypothesis, we have

$$\Box\varphi_1 \rightarrow (\Box\varphi_2 \rightarrow \cdots \Box(\varphi_{n-1} \rightarrow \varphi_n) \cdots) \in \Sigma$$

Since Σ is a normal modal logic, it contains all instances of **K**, in particular

$$\Box(\varphi_{n-1} \rightarrow \varphi_n) \rightarrow (\Box\varphi_{n-1} \rightarrow \Box\varphi_n) \in \Sigma$$

Using modus ponens and suitable tautological instances we get

$$\Box\varphi_1 \rightarrow (\Box\varphi_2 \rightarrow \cdots (\Box\varphi_{n-1} \rightarrow \Box\varphi_n) \cdots) \in \Sigma. \quad \square$$

Proposition prf.4. *Every normal modal logic Σ contains $\neg\Diamond\perp$.*

*nml:prf:nor:
prop:notDiamondBot*

Problem prf.1. Prove **Proposition prf.4**.

Proposition prf.5. *Let $\varphi_1, \dots, \varphi_n$ be **formulas**. Then there is a smallest modal logic Σ containing all instances of $\varphi_1, \dots, \varphi_n$.*

Proof. Given $\varphi_1, \dots, \varphi_n$, define Σ as the intersection of all normal modal logics containing all instances of $\varphi_1, \dots, \varphi_n$. The intersection is non-empty as $\text{Frm}(\mathcal{L})$, the set of all **formulas**, is such a modal logic. \square

Definition prf.6. The smallest normal modal logic containing $\varphi_1, \dots, \varphi_n$ is called a *modal system* and denoted by $\mathbf{K}\varphi_1 \dots \varphi_n$. The smallest normal modal logic is denoted by **K**.

Photo Credits

Bibliography