

## prf.1 Derivations and Modal Systems

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We first define what a **derivation** is for normal modal logics. Roughly, a **derivation** is a sequence of **formulas** in which every **element** is either (a substitution instance of) one of a number of *axioms*, or follows from previous **elements** by one of a few inference rules. For normal modal logics, all instances of tautologies, K, and DUAL count as axioms. This results in the modal system **K**, the smallest normal modal logic. We may wish to add additional axioms to obtain other systems, however. The rules are always modus ponens MP and necessitation NEC.

**Definition prf.1.** Given a modal system  $\mathbf{K}\varphi_1 \dots \varphi_n$  and a **formula**  $\psi$  we say that  $\psi$  is **derivable** in  $\mathbf{K}\varphi_1 \dots \varphi_n$ , written  $\mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi$ , if and only if there are **formulas**  $\chi_1, \dots, \chi_k$  such that  $\chi_k = \psi$  and each  $\chi_i$  is either a tautological instance, or an instance of one of K, DUAL,  $\varphi_1, \dots, \varphi_n$ , or it follows from previous **formulas** by means of the rules MP or NEC.

The following proposition allows us to show that  $\psi \in \Sigma$  by exhibiting a  $\Sigma$ -proof of  $\psi$ .

**Proposition prf.2.**  $\mathbf{K}\varphi_1 \dots \varphi_n = \{\psi : \mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi\}$ .

*Proof.* We use induction on the length of **derivations** to show that  $\{\psi : \mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi\} \subseteq \mathbf{K}\varphi_1 \dots \varphi_n$ .

If the **derivation** of  $\psi$  has length 1, it contains a single **formula**. That **formula** cannot follow from previous formulas by MP or NEC, so must be a tautological instance, an instance of K, DUAL, or an instance of one of  $\varphi_1, \dots, \varphi_n$ . But  $\mathbf{K}\varphi_1 \dots \varphi_n$  contains these as well, so  $\psi \in \mathbf{K}\varphi_1 \dots \varphi_n$ .

If the **derivation** of  $\psi$  has length  $> 1$ , then  $\psi$  may in addition be obtained by MP or NEC from **formulas** not occurring as the last line in the **derivation**. If  $\psi$  follows from  $\chi$  and  $\chi \rightarrow \psi$  (by MP), then  $\chi$  and  $\chi \rightarrow \psi \in \mathbf{K}\varphi_1 \dots \varphi_n$  by induction hypothesis. But every modal logic is closed under modus ponens, so  $\psi \in \mathbf{K}\varphi_1 \dots \varphi_n$ . If  $\psi \equiv \Box\chi$  follows from  $\chi$  by NEC, then  $\chi \in \mathbf{K}\varphi_1 \dots \varphi_n$  by induction hypothesis. But every normal modal logic is closed under NEC, so  $\psi \in \mathbf{K}\varphi_1 \dots \varphi_n$ .

The converse inclusion follows by showing that  $\Sigma = \{\psi : \mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi\}$  is a normal modal logic containing all the instances of  $\varphi_1, \dots, \varphi_n$ , and the observation that  $\mathbf{K}\varphi_1 \dots \varphi_n$  is, by definition, the smallest such logic.

1. Every tautology  $\psi$  is a tautological instance, so  $\mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi$ , so  $\Sigma$  contains all tautologies.
2. If  $\mathbf{K}\varphi_1 \dots \varphi_n \vdash \chi$  and  $\mathbf{K}\varphi_1 \dots \varphi_n \vdash \chi \rightarrow \psi$ , then  $\mathbf{K}\varphi_1 \dots \varphi_n \vdash \psi$ : Combine the **derivation** of  $\chi$  with that of  $\chi \rightarrow \psi$ , and add the line  $\psi$ . The last line is justified by MP. So  $\Sigma$  is closed under modus ponens.
3. If  $\psi$  has a **derivation**, then every substitution instance of  $\psi$  also has a derivation: apply the substitution to every **formula** in the **derivation**.

(Exercise: prove by induction on the length of **derivations** that the result is also a correct **derivation**). So  $\Sigma$  is closed under uniform substitution. (We have now established that  $\Sigma$  satisfies all conditions of a modal logic.)

4. We have  $\mathbf{K}\varphi_1 \dots \varphi_n \vdash K$ , so  $K \in \Sigma$ .
5. We have  $\mathbf{K}\varphi_1 \dots \varphi_n \vdash DUAL$ , so  $DUAL \in \Sigma$ .
6. If  $\mathbf{K}\varphi_1 \dots \varphi_n \vdash \chi$ , the additional line  $\Box\chi$  is justified by NEC. Consequently,  $\Sigma$  is closed under NEC. Thus,  $\Sigma$  is normal.  $\square$

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## Bibliography