prf.1 Derived Rules

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Finding and writing derivations is obviously difficult, cumbersome, and repetitive. For instance, very often we want to pass from $\varphi \rightarrow \psi$ to $\Box\varphi \rightarrow \Box\psi$, i.e., apply rule RK. That requires an application of NEC, then recording the proper instance of K, then applying MP. Passing from $\varphi \rightarrow \psi$ and $\psi \rightarrow \chi$ to $\varphi \rightarrow \chi$ requires recording the (long) tautological instance

$$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$$

and applying MP twice. Often we want to replace a sub-formula by a formula we know to be equivalent, e.g., $\Diamond\varphi$ by $\neg\Box\neg\varphi$, or $\neg\neg\varphi$ by φ. So rather than write out the actual derivation, it is more convenient to simply record why the intermediate steps are derivable. For this purpose, let us collect some facts about derivability.

Proposition prf.1. *If* $\mathbf{K} \vdash \varphi_1$*,* $\dots$*,* $\mathbf{K} \vdash \varphi_n$*, and* ψ *follows from* $\varphi_1, \dots, \varphi_n$ *by propositional logic, then* $\mathbf{K} \vdash \psi$*.*

Proof. If ψ follows from $\varphi_1, \dots, \varphi_n$ by propositional logic, then

$$\varphi_1 \rightarrow (\varphi_2 \rightarrow \cdots (\varphi_n \rightarrow \psi) \dots)$$

is a tautological instance. Applying MP n times gives a derivation of ψ. □

We will indicate use of this proposition by PL.

Proposition prf.2. *If* $\mathbf{K} \vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow \cdots (\varphi_{n-1} \rightarrow \varphi_n) \dots)$ *then* $\mathbf{K} \vdash \Box\varphi_1 \rightarrow (\Box\varphi_2 \rightarrow \cdots (\Box\varphi_{n-1} \rightarrow \Box\varphi_n) \dots)$*.*

Proof. By induction on n, just as in the proof of **??**. □

We will indicate use of this proposition by RK. Let's illustrate how these results help establishing derivability results more easily.

Proposition prf.3. $\mathbf{K} \vdash (\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

Proof.

1.	$\mathbf{K} \vdash \varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$	TAUT
2.	$\mathbf{K} \vdash \Box\varphi \rightarrow (\Box\psi \rightarrow \Box(\varphi \wedge \psi)))$	RK, 1
3.	$\mathbf{K} \vdash (\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$	PL, 2

□

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Proposition prf.4. *If* $\mathbf{K} \vdash \varphi \leftrightarrow \psi$ *and* $\mathbf{K} \vdash \chi[\varphi/q]$ *then* $\mathbf{K} \vdash \chi[B/q]$

Proof. Exercise. □

Problem prf.1. Prove Proposition prf.4 by proving, by induction on the complexity of χ, that if $\mathbf{K} \vdash \varphi \leftrightarrow \psi$ then $\mathbf{K} \vdash \chi[\varphi/q] \leftrightarrow \chi[\psi/q]$.

This proposition comes in handy especially when we want to convert $\Diamond$ into $\Box$ (or vice versa), or remove double negations inside a formula. In what follows, we will mark applications of Proposition prf.4 by "φ for ψ" whenever we re-write a formula $\chi(\psi)$ for $\chi(\varphi)$. In other words, "φ for ψ" abbreviates:

$\vdash \chi(\varphi)$	
$\vdash \varphi \leftrightarrow \psi$	
$\vdash \chi(\psi)$	by Proposition prf.4

For instance:

Proposition prf.5. $\mathbf{K} \vdash \neg\Box p \to \Diamond\neg p$

Proof.

1.	$\mathbf{K} \vdash \Diamond\neg p \leftrightarrow \neg\Box\neg\neg p$	DUAL
2.	$\mathbf{K} \vdash \neg\Box\neg\neg p \to \Diamond\neg p$	PL, 1
3.	$\mathbf{K} \vdash \neg\Box p \to \Diamond\neg p$	p for $\neg\neg p$

□

In the above derivation, the final step "p for $\neg\neg p$" is short for

$\mathbf{K} \vdash \neg\Box\neg\neg p \to \Diamond\neg p$	
$\mathbf{K} \vdash \neg\neg p \leftrightarrow p$	TAUT
$\mathbf{K} \vdash \neg\Box p \to \Diamond\neg p$	by Proposition prf.4

The roles of $\chi(q)$, φ, and ψ in Proposition prf.4 are played here, respectively, by $\neg\Box q \to \Diamond\neg p$, $\neg\neg p$, and p.

When a formula contains a sub-formula $\neg\Diamond\varphi$, we can replace it by $\Box\neg\varphi$ using Proposition prf.4, since $\mathbf{K} \vdash \neg\Diamond\varphi \leftrightarrow \Box\neg\varphi$. We'll indicate this and similar replacements simply by "$\Box\neg$ for $\neg\Diamond$."

The following proposition justifies that we can establish derivability results schematically. E.g., the previous proposition does not just establish that $\mathbf{K} \vdash \neg\Box p \to \Diamond\neg p$, but $\mathbf{K} \vdash \neg\Box\varphi \to \Diamond\neg\varphi$ for arbitrary φ.

Proposition prf.6. *If φ is a substitution instance of ψ and $\mathbf{K} \vdash \psi$, then $\mathbf{K} \vdash \varphi$.*

Proof. It is tedious but routine to verify (by induction on the length of the derivation of ψ) that applying a substitution to an entire derivation also results in a correct derivation. Specifically, substitution instances of tautological instances are themselves tautological instances, substitution instances of instances of DUAL and K are themselves instances of DUAL and K, and applications of MP and NEC remain correct when substituting formulas for propositional variables in both premise(s) and conclusion. □

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Bibliography