prf.1 Consistency

nml:prf:con: sec

Consistency is an important property of sets of formulas. A set of formulas is inconsistent if a contradiction, such as $\bot$, is derivable from it; and otherwise consistent. If a set is inconsistent, its formulas cannot all be true in a model at a world. For the completeness theorem we prove the converse: every consistent set is true at a world in a model, namely in the "canonical model."

Definition prf.1. A set Γ is *consistent* relatively to a system Σ or, as we will say, Σ-consistent, if and only if $\Gamma \nvdash_{\Sigma} \bot$.

So for instance, the set $\{\Box(p \to q), \Box p, \neg\Box q\}$ is consistent relatively to propositional logic, but not **K**-consistent. Similarly, the set $\{\Diamond p, \Box\Diamond p \to q, \neg q\}$ is not **K5**-consistent.

nml:prf:con: prop:consistencyfacts

Proposition prf.2. *Let Γ be a set of formulas. Then:*

1. *A set Γ is Σ-consistent if and only if there is some formula φ such that $\Gamma \nvdash_{\Sigma} \varphi$.*
2. *nml:prf:con: prop:consistencyfacts-b* *$\Gamma \vdash_{\Sigma} \varphi$ if and only if $\Gamma \cup \{\neg\varphi\}$ is not Σ-consistent.*
3. *nml:prf:con: prop:consistencyfacts-c* *If Γ is Σ-consistent, then for any formula φ, either $\Gamma \cup \{\varphi\}$ is Σ-consistent or $\Gamma \cup \{\neg\varphi\}$ is Σ-consistent.*

Proof. These facts follow easily using classical propositional logic. We give the argument for (3). Proceed contrapositively and suppose neither $\Gamma \cup \{\varphi\}$ nor $\Gamma \cup \{\neg\varphi\}$ is Σ-consistent. Then by (2), both $\Gamma, \varphi \vdash_{\Sigma} \bot$ and $\Gamma, \neg\varphi \vdash_{\Sigma} \bot$. By the deduction theorem $\Gamma \vdash_{\Sigma} \varphi \to \bot$ and $\Gamma \vdash_{\Sigma} \neg\varphi \to \bot$. But $(\varphi \to \bot) \to ((\neg\varphi \to \bot) \to \bot)$ is a tautological instance, hence by **????**, $\Gamma \vdash_{\Sigma} \bot$. $\Box$

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Bibliography