Consistency is an important property of sets of formulas. A set of formulas is inconsistent if a contradiction, such as $\bot$, is derivable from it; and otherwise consistent. If a set is inconsistent, its formulas cannot all be true in a model at a world. For the completeness theorem we prove the converse: every consistent set is true at a world in a model, namely in the “canonical model.”

**Definition prf.1.** A set $\Gamma$ is consistent relatively to a system $\Sigma$ or, as we will say, $\Sigma$-consistent, if and only if $\Gamma \nvdash_\Sigma \bot$.

So for instance, the set \{\Box(p \to q), \Box p, \neg \Box q\} is consistent relatively to propositional logic, but not $K$-consistent. Similarly, the set \{\Diamond p, \Box \Diamond p \to q, \neg q\} is not $K5$-consistent.

**Proposition prf.2.** Let $\Gamma$ be a set of formulas. Then:

1. $\Gamma$ is $\Sigma$-consistent if and only if there is some formula $\varphi$ such that $\Gamma \nvdash_\Sigma \varphi$.
2. $\Gamma \vdash_\Sigma \varphi$ if and only if $\Gamma \cup \{\neg \varphi\}$ is not $\Sigma$-consistent.
3. If $\Gamma$ is $\Sigma$-consistent, then for any formula $\varphi$, either $\Gamma \cup \{\varphi\}$ is $\Sigma$-consistent or $\Gamma \cup \{\neg \varphi\}$ is $\Sigma$-consistent.

**Proof.** These facts follow easily using classical propositional logic. We give the argument for (3). Proceed contrapositively and suppose neither $\Gamma \cup \{\varphi\}$ nor $\Gamma \cup \{\neg \varphi\}$ is $\Sigma$-consistent. Then by (2), both $\Gamma, \varphi \vdash_\Sigma \bot$ and $\Gamma, \neg \varphi \vdash_\Sigma \bot$. By the deduction theorem $\Gamma \vdash_\Sigma \varphi \to \bot$ and $\Gamma \vdash_\Sigma \neg \varphi \to \bot$. But $(\varphi \to \bot) \to ((\neg \varphi \to \bot) \to \bot)$ is a tautological instance, hence by $???$, $\Gamma \vdash_\Sigma \bot$. $\square$

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**Bibliography**