Chapter udf

Models of Arithmetic

mar.1 Introduction

The *standard model* of arithmetic is the structure $\mathfrak{N}$ with $|\mathfrak{N}| = \mathbb{N}$ in which o, $\prime$, $+$, $\times$, and $<$ are interpreted as you would expect. That is, o is 0, $\prime$ is the successor function, $+$ is interpreted as addition and $\times$ as multiplication of the numbers in $\mathbb{N}$. Specifically,

$$\begin{aligned} \mathsf{o}^{\mathfrak{N}} &= 0 \\ \prime^{\mathfrak{N}}(n) &= n + 1 \\ +^{\mathfrak{N}}(n, m) &= n + m \\ \times^{\mathfrak{N}}(n, m) &= nm \end{aligned}$$

Of course, there are structures for $\mathcal{L}_A$ that have domains other than $\mathbb{N}$. For instance, we can take $\mathfrak{M}$ with domain $|\mathfrak{M}| = \{a\}^*$ (the finite sequences of the single symbol a, i.e., $\emptyset$, a, aa, aaa, ...), and interpretations

$$\begin{aligned} \mathsf{o}^{\mathfrak{M}} &= \emptyset \\ \prime^{\mathfrak{M}}(s) &= s \frown a \\ +^{\mathfrak{M}}(n, m) &= a^{n+m} \\ \times^{\mathfrak{M}}(n, m) &= a^{nm} \end{aligned}$$

These two structures are "essentially the same" in the sense that the only difference is the elements of the domains but not how the elements of the domains are related among each other by the interpretation functions. We say that the two structures are *isomorphic*.

It is an easy consequence of the compactness theorem that any theory true in $\mathfrak{N}$ also has models that are not isomorphic to $\mathfrak{N}$. Such structures are called *non-standard*. The interesting thing about them is that while the elements of a standard model (i.e., $\mathfrak{N}$, but also all structures isomorphic to it) are exhausted by the values of the standard numerals $\overline{n}$, i.e.,

$$|\mathfrak{N}| = \{\mathrm{Val}^{\mathfrak{N}}(\overline{n}) : n \in \mathbb{N}\}$$

that isn't the case in non-standard models: if $\mathfrak{M}$ is non-standard, then there is at least one $x \in |\mathfrak{M}|$ such that $x \neq \mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for all n.

These non-standard elements are pretty neat: they are "infinite natural numbers." But their existence also explains, in a sense, the incompleteness phenomena. Consider an example, e.g., the consistency statement for Peano arithmetic, $\mathsf{Con}_{\mathbf{PA}}$, i.e., $\lnot\exists x\, \mathsf{Prf}_{\mathbf{PA}}(x, \ulcorner\bot\urcorner)$. Since $\mathbf{PA}$ neither proves $\mathsf{Con}_{\mathbf{PA}}$ nor $\lnot\mathsf{Con}_{\mathbf{PA}}$, either can be consistently added to $\mathbf{PA}$. Since $\mathbf{PA}$ is consistent, $\mathfrak{N} \vDash \mathsf{Con}_{\mathbf{PA}}$, and consequently $\mathfrak{N} \nvDash \lnot\mathsf{Con}_{\mathbf{PA}}$. So $\mathfrak{N}$ is *not* a model of $\mathbf{PA} \cup \{\lnot\mathsf{Con}_{\mathbf{PA}}\}$, and all its models must be nonstandard. Models of $\mathbf{PA} \cup \{\lnot\mathsf{Con}_{\mathbf{PA}}\}$ must contain some element that serves as the witness that makes $\exists x\, \mathsf{Prf}_{\mathbf{PA}}(\ulcorner\bot\urcorner)$ true, i.e., a Gödel number of a derivation of a contradiction from $\mathbf{PA}$. Such an element can't be standard—since $\mathbf{PA} \vdash \lnot\mathsf{Prf}_{\mathbf{PA}}(\overline{n}, \ulcorner\bot\urcorner)$ for every n.

mar.2 Standard Models of Arithmetic

The language of arithmetic $\mathcal{L}_A$ is obviously intended to be about numbers, specifically, about natural numbers. So, "the" standard model $\mathfrak{N}$ is special: it is the model we want to talk about. But in logic, we are often just interested in structural properties, and any two structures that are isomorphic share those. So we can be a bit more liberal, and consider any structure that is isomorphic to $\mathfrak{N}$ "standard." mod:mar:stm:sec

Definition mar.1. A structure for $\mathcal{L}_A$ is *standard* if it is isomorphic to $\mathfrak{N}$.

Proposition mar.2. *If a structure $\mathfrak{M}$ is standard, then its domain is the set of values of the standard numerals, i.e.,* mod:mar:stm:prop:standard-domain

$$|\mathfrak{M}| = \{\mathrm{Val}^{\mathfrak{M}}(\overline{n}) : n \in \mathbb{N}\}$$

Proof. Clearly, every $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) \in |\mathfrak{M}|$. We just have to show that every $x \in |\mathfrak{M}|$ is equal to $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for some n. Since $\mathfrak{M}$ is standard, it is isomorphic to $\mathfrak{N}$. Suppose $g\colon \mathbb{N} \to |\mathfrak{M}|$ is an isomorphism. Then $g(n) = g(\mathrm{Val}^{\mathfrak{N}}(\overline{n})) = \mathrm{Val}^{\mathfrak{M}}(\overline{n})$. But for every $x \in |\mathfrak{M}|$, there is an $n \in \mathbb{N}$ such that $g(n) = x$, since g is surjective. □

explanation If a structure $\mathfrak{M}$ for $\mathcal{L}_A$ is standard, the elements of its domain can all be named by the standard numerals $\overline{0}$, $\overline{1}$, $\overline{2}$, ..., i.e., the terms o, o', o'', etc. Of course, this does not mean that the elements of $|\mathfrak{M}|$ *are* the numbers, just that we can pick them out the same way we can pick out the numbers in $|\mathfrak{N}|$.

Problem mar.1. Show that the converse of Proposition mar.2 is false, i.e., give an example of a structure $\mathfrak{M}$ with $|\mathfrak{M}| = \{\mathrm{Val}^{\mathfrak{M}}(\overline{n}) : n \in \mathbb{N}\}$ that is not isomorphic to $\mathfrak{N}$.

Proposition mar.3. *If $\mathfrak{M} \vDash \mathbf{Q}$, and $|\mathfrak{M}| = \{\mathrm{Val}^{\mathfrak{M}}(\overline{n}) : n \in \mathbb{N}\}$, then $\mathfrak{M}$ is standard.* mod:mar:stm:prop:thq-standard

Proof. We have to show that $\mathfrak{M}$ is isomorphic to $\mathfrak{N}$. Consider the function $g\colon \mathbb{N} \to |\mathfrak{M}|$ defined by $g(n) = \mathrm{Val}^{\mathfrak{M}}(\overline{n})$. By the hypothesis, g is surjective. It is also injective: $\mathbf{Q} \vdash \overline{n} \neq \overline{m}$ whenever $n \neq m$. Thus, since $\mathfrak{M} \vDash \mathbf{Q}$, $\mathfrak{M} \vDash \overline{n} \neq \overline{m}$, whenever $n \neq m$. Thus, if $n \neq m$, then $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) \neq \mathrm{Val}^{\mathfrak{M}}(\overline{m})$, i.e., $g(n) \neq g(m)$.

We also have to verify that g is an isomorphism.

1. We have $g(\mathsf{o}^{\mathfrak{N}}) = g(0)$ since, $\mathsf{o}^{\mathfrak{N}} = 0$. By definition of g, $g(0) = \mathrm{Val}^{\mathfrak{M}}(\overline{0})$. But $\overline{0}$ is just o, and the value of a term which happens to be a constant symbol is given by what the structure assigns to that constant symbol, i.e., $\mathrm{Val}^{\mathfrak{M}}(\mathsf{o}) = \mathsf{o}^{\mathfrak{M}}$. So we have $g(\mathsf{o}^{\mathfrak{N}}) = \mathsf{o}^{\mathfrak{M}}$ as required.

2. $g({\prime}^{\mathfrak{N}}(n)) = g(n+1)$, since $\prime$ in $\mathfrak{N}$ is the successor function on $\mathbb{N}$. Then, $g(n+1) = \mathrm{Val}^{\mathfrak{M}}(\overline{n+1})$ by definition of g. But $\overline{n+1}$ is the same term as $\overline{n}'$, so $\mathrm{Val}^{\mathfrak{M}}(\overline{n+1}) = \mathrm{Val}^{\mathfrak{M}}(\overline{n}')$. By the definition of the value function, this is $= {\prime}^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}(\overline{n}))$. Since $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) = g(n)$ we get $g({\prime}^{\mathfrak{N}}(n)) = {\prime}^{\mathfrak{M}}(g(n))$.

3. $g(+^{\mathfrak{N}}(n,m)) = g(n+m)$, since $+$ in $\mathfrak{N}$ is the addition function on $\mathbb{N}$. Then, $g(n+m) = \mathrm{Val}^{\mathfrak{M}}(\overline{n+m})$ by definition of g. But $\mathbf{Q} \vdash \overline{n+m} = (\overline{n} + \overline{m})$, so $\mathrm{Val}^{\mathfrak{M}}(\overline{n+m}) = \mathrm{Val}^{\mathfrak{M}}(\overline{n} + \overline{m})$. By the definition of the value function, this is $= +^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}(\overline{n}), \mathrm{Val}^{\mathfrak{M}}(\overline{m}))$. Since $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) = g(n)$ and $\mathrm{Val}^{\mathfrak{M}}(\overline{m}) = g(m)$, we get $g(+^{\mathfrak{N}}(n,m)) = +^{\mathfrak{M}}(g(n), g(m))$.

4. $g(\times^{\mathfrak{N}}(n,m)) = \times^{\mathfrak{M}}(g(n), g(m))$: Exercise.

5. $\langle n, m \rangle \in <^{\mathfrak{N}}$ iff $n < m$. If $n < m$, then $\mathbf{Q} \vdash \overline{n} < \overline{m}$, and also $\mathfrak{M} \vDash \overline{n} < \overline{m}$. Thus $\langle \mathrm{Val}^{\mathfrak{M}}(\overline{n}), \mathrm{Val}^{\mathfrak{M}}(\overline{m}) \rangle \in <^{\mathfrak{M}}$, i.e., $\langle g(n), g(m) \rangle \in <^{\mathfrak{M}}$. If $n \not< m$, then $\mathbf{Q} \vdash \lnot \overline{n} < \overline{m}$, and consequently $\mathfrak{M} \nvDash \overline{n} < \overline{m}$. Thus, as before, $\langle g(n), g(m) \rangle \notin <^{\mathfrak{M}}$. Together, we get: $\langle n, m \rangle \in <^{\mathfrak{N}}$ iff $\langle g(n), g(m) \rangle \in <^{\mathfrak{M}}$. □

The function g is the most obvious way of defining a mapping from $\mathbb{N}$ to the domain of any other structure $\mathfrak{M}$ for $\mathcal{L}_A$, since every such $\mathfrak{M}$ contains elements named by $\overline{0}$, $\overline{1}$, $\overline{2}$, etc. So it isn't surprising that if $\mathfrak{M}$ makes at least some basic statements about the $\overline{n}$'s true in the same way that $\mathfrak{N}$ does, and g is also bijective, then g will turn into an isomorphism. In fact, if $|\mathfrak{M}|$ contains no elements other than what the $\overline{n}$'s name, it's the only one. explanation

mod:mar:stm: prop:thq-unique-iso **Proposition mar.4.** *If $\mathfrak{M}$ is standard, then g from the proof of Proposition mar.3 is the only isomorphism from $\mathfrak{N}$ to $\mathfrak{M}$.*

Proof. Suppose $h\colon \mathbb{N} \to |\mathfrak{M}|$ is an isomorphism between $\mathfrak{N}$ and $\mathfrak{M}$. We show that $g = h$ by induction on n. If $n = 0$, then $g(0) = \mathsf{o}^{\mathfrak{M}}$ by definition of g. But since h is an isomorphism, $h(0) = h(\mathsf{o}^{\mathfrak{N}}) = \mathsf{o}^{\mathfrak{M}}$, so $g(0) = h(0)$.

Now consider the case for $n+1$. We have

$$\begin{aligned}
g(n+1) &= \mathrm{Val}^{\mathfrak{M}}(\overline{n+1}) \text{ by definition of } g\\
&= \mathrm{Val}^{\mathfrak{M}}(\overline{n}') \text{ since } \overline{n+1} \equiv \overline{n}'\\
&= {\prime}^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}(\overline{n})) \text{ by definition of } \mathrm{Val}^{\mathfrak{M}}(t')\\
&= {\prime}^{\mathfrak{M}}(g(n)) \text{ by definition of } g\\
&= {\prime}^{\mathfrak{M}}(h(n)) \text{ by induction hypothesis}\\
&= h({\prime}^{\mathfrak{N}}(n)) \text{ since } h \text{ is an isomorphism}\\
&= h(n+1)
\end{aligned}$$

□

explanation For any denumerable set M, there's a bijection between $\mathbb{N}$ and M, so every such set M is potentially the domain of a standard model $\mathfrak{M}$. In fact, once you pick an object $z \in M$ and a suitable function s as $\mathsf{o}^{\mathfrak{M}}$ and ${\prime}^{\mathfrak{M}}$, the interpretations of $+$, $\times$, and $<$ is already fixed. Only functions $s\colon M \to M \setminus \{z\}$ that are both injective and surjective are suitable in a standard model as ${\prime}^{\mathfrak{M}}$. The range of s cannot contain z, since otherwise $\forall x\, \mathsf{o} \neq x'$ would be false. That sentence is true in $\mathfrak{N}$, and so $\mathfrak{M}$ also has to make it true. The function s has to be injective, since the successor function ${\prime}^{\mathfrak{N}}$ in $\mathfrak{N}$ is, and that ${\prime}^{\mathfrak{N}}$ is injective is expressed by a sentence true in $\mathfrak{N}$. It has to be surjective because otherwise there would be some $x \in M \setminus \{z\}$ not in the domain of s, i.e., the sentence $\forall x\,(x = \mathsf{o} \lor \exists y\, y' = x)$ would be false in $\mathfrak{M}$—but it is true in $\mathfrak{N}$.

mar.3 Non-Standard Models

explanation We call a structure for $\mathcal{L}_A$ standard if it is isomorphic to $\mathfrak{N}$. If a structure isn't isomorphic to $\mathfrak{N}$, it is called non-standard.

Definition mar.5. A structure $\mathfrak{M}$ for $\mathcal{L}_A$ is *non-standard* if it is not isomorphic to $\mathfrak{N}$. The elements $x \in |\mathfrak{M}|$ which are equal to $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for some $n \in \mathbb{N}$ are called *standard numbers* (of $\mathfrak{M}$), and those not, *non-standard numbers*.

explanation By Proposition mar.2, any standard structure for $\mathcal{L}_A$ contains only standard elements. Consequently, a non-standard structure must contain at least one non-standard element. In fact, the existence of a non-standard element guarantees that the structure is non-standard.

Proposition mar.6. *If a structure $\mathfrak{M}$ for $\mathcal{L}_A$ contains a non-standard number, $\mathfrak{M}$ is non-standard.*

Proof. Suppose not, i.e., suppose $\mathfrak{M}$ standard but contains a non-standard number x. Let $g\colon \mathbb{N} \to |\mathfrak{M}|$ be an isomorphism. It is easy to see (by induction on n) that $g(\mathrm{Val}^{\mathfrak{N}}(\overline{n})) = \mathrm{Val}^{\mathfrak{M}}(\overline{n})$. In other words, g maps standard numbers of $\mathfrak{N}$ to standard numbers of $\mathfrak{M}$. If $\mathfrak{M}$ contains a non-standard number, g cannot be surjective, contrary to hypothesis. □

Problem mar.2. Recall that $\mathbf{Q}$ contains the axioms

$$\forall x\,\forall y\,(x' = y' \rightarrow x = y) \tag{Q_1}$$

$$\forall x\,\mathsf{o} \neq x' \tag{Q_2}$$

$$\forall x\,(x = \mathsf{o} \lor \exists y\, x = y') \tag{Q_3}$$

Give structures $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$ such that

1. $\mathfrak{M}_1 \vDash Q_1$, $\mathfrak{M}_1 \vDash Q_2$, $\mathfrak{M}_1 \nvDash Q_3$;

2. $\mathfrak{M}_2 \vDash Q_1$, $\mathfrak{M}_2 \nvDash Q_2$, $\mathfrak{M}_2 \vDash Q_3$; and

3. $\mathfrak{M}_3 \nvDash Q_1$, $\mathfrak{M}_3 \vDash Q_2$, $\mathfrak{M}_3 \vDash Q_3$;

Obviously, you just have to specify $\mathsf{o}^{\mathfrak{M}_i}$ and $\prime^{\mathfrak{M}_i}$ for each.

It is easy enough to specify non-standard structures for $\mathcal{L}_A$. For instance, take the structure with domain $\mathbb{Z}$ and interpret all non-logical symbols as usual. Since negative numbers are not values of $\overline{n}$ for any n, this structure is non-standard. Of course, it will not be a *model* of arithmetic in the sense that it makes the same sentences true as $\mathfrak{N}$. For instance, $\forall x\, x' \neq \mathsf{o}$ is false. However, we can prove that non-standard models of arithmetic exist easily enough, using the compactness theorem. explanation

Proposition mar.7. *Let* $\mathbf{TA} = \{\varphi : \mathfrak{N} \vDash \varphi\}$ *be the theory of* $\mathfrak{N}$. $\mathbf{TA}$ *has an enumerable non-standard model.*

Proof. Expand $\mathcal{L}_A$ by a new constant symbol c and consider the set of sentences

$$\Gamma = \mathbf{TA} \cup \{c \neq \overline{0}, c \neq \overline{1}, c \neq \overline{2}, \dots\}$$

Any model $\mathfrak{M}^c$ of Γ would contain an element $x = c^{\mathfrak{M}}$ which is non-standard, since $x \neq \mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for all $n \in \mathbb{N}$. Also, obviously, $\mathfrak{M}^c \vDash \mathbf{TA}$, since $\mathbf{TA} \subseteq \Gamma$. If we turn $\mathfrak{M}^c$ into a structure $\mathfrak{M}$ for $\mathcal{L}_A$ simply by forgetting about c, its domain still contains the non-standard x, and also $\mathfrak{M} \vDash \mathbf{TA}$. The latter is guaranteed since c does not occur in $\mathbf{TA}$. So, it suffices to show that Γ has a model.

We use the compactness theorem to show that Γ has a model. If every finite subset of Γ is satisfiable, so is Γ. Consider any finite subset $\Gamma_0 \subseteq \Gamma$. Γ_0 includes some sentences of $\mathbf{TA}$ and some of the form $c \neq \overline{n}$, but only finitely many. Suppose k is the largest number so that $c \neq \overline{k} \in \Gamma_0$. Define $\mathfrak{N}_k$ by expanding $\mathfrak{N}$ to include the interpretation $c^{\mathfrak{N}_k} = k + 1$. $\mathfrak{N}_k \vDash \Gamma_0$: if $\varphi \in \mathbf{TA}$, $\mathfrak{N}_k \vDash \varphi$ since $\mathfrak{N}_k$ is just like $\mathfrak{N}$ in all respects except c, and c does not occur in φ. And $\mathfrak{N}_k \vDash c \neq \overline{n}$, since $n \leq k$, and $\mathrm{Val}^{\mathfrak{N}_k}(c) = k + 1$. Thus, every finite subset of Γ is satisfiable. □

mar.4 Models of Q

explanation

We know that there are non-standard structures that make the same sentences true as $\mathfrak{N}$ does, i.e., is a model of **TA**. Since $\mathfrak{N} \vDash \mathbf{Q}$, any model of **TA** is also a model of **Q**. **Q** is much weaker than **TA**, e.g., $\mathbf{Q} \nvDash \forall x \, \forall y \, (x + y) = (y + x)$. Weaker theories are easier to satisfy: they have more models. E.g., **Q** has models which make $\forall x \, \forall y \, (x + y) = (y + x)$ false, but those cannot also be models of **TA**, or **PA** for that matter. Models of **Q** are also relatively simple: we can specify them explicitly.

Example mar.8. Consider the structure $\mathfrak{K}$ with domain $|\mathfrak{K}| = \mathbb{N} \cup \{a\}$ and interpretations mod:mar:mdq:ex:model-K-of-Q

$$\begin{aligned}
\mathsf{o}^{\mathfrak{K}} &= 0 \\
{\prime}^{\mathfrak{K}}(x) &= \begin{cases} x+1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases} \\
+^{\mathfrak{K}}(x, y) &= \begin{cases} x+y & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\
\times^{\mathfrak{K}}(x, y) &= \begin{cases} xy & \text{if } x, y \in \mathbb{N} \\ 0 & \text{if } x = 0 \text{ or } y = 0 \\ a & \text{otherwise} \end{cases} \\
<^{\mathfrak{K}} &= \{\langle x, y \rangle : x, y \in \mathbb{N} \text{ and } x < y\} \cup \{\langle x, a \rangle : x \in |\mathfrak{K}|\}
\end{aligned}$$

To show that $\mathfrak{K} \vDash \mathbf{Q}$ we have to verify that all axioms of **Q** are true in $\mathfrak{K}$. For convenience, let's write x^* for ${\prime}^{\mathfrak{K}}(x)$ (the "successor" of x in $\mathfrak{K}$), $x \oplus y$ for $+^{\mathfrak{K}}(x, y)$ (the "sum" of x and y in $\mathfrak{K}$, $x \otimes y$ for $\times^{\mathfrak{K}}(x, y)$ (the "product" of x and y in $\mathfrak{K}$), and $x \otimes y$ for $\langle x, y \rangle \in <^{\mathfrak{K}}$. With these abbreviations, we can give the operations in $\mathfrak{K}$ more perspicuously as

x	x^*
n	$n+1$
a	a

$x \oplus y$	0	m	a
0	0	m	a
n	n	$n+m$	a
a	a	a	a

$x \otimes y$	0	m	a
0	0	0	0
n	0	nm	a
a	0	a	a

We have $n \otimes m$ iff $n < m$ for $n, m \in \mathbb{N}$ and $x \otimes a$ for all $x \in |\mathfrak{K}|$.

$\mathfrak{K} \vDash \forall x \, \forall y \, (x' = y' \rightarrow x = y)$ since $*$ is injective. $\mathfrak{K} \vDash \forall x \, \mathsf{o} \neq x'$ since 0 is not a $*$-successor in $\mathfrak{K}$. $\mathfrak{K} \vDash \forall x \, (x = \mathsf{o} \lor \exists y \, x = y')$ since for every $n > 0$, $n = (n-1)^*$, and $a = a^*$.

$\mathfrak{K} \vDash \forall x \, (x + \mathsf{o}) = x$ since $n \oplus 0 = n + 0 = n$, and $a \oplus 0 = a$ by definition of $\oplus$. $\mathfrak{K} \vDash \forall x \, \forall y \, (x + y') = (x + y)'$ is a bit trickier. If n, m are both standard, we have:

$$(n \oplus m^*) = (n + (m + 1)) = (n + m) + 1 = (n \oplus m)^*$$

since $\oplus$ and * agree with $+$ and $\prime$ on standard numbers. Now suppose $x \in |\mathfrak{K}|$. Then

$$(x \oplus a^*) = (x \oplus a) = a = a^* = (x \oplus a)^*$$

The remaining case is if $y \in |\mathfrak{K}|$ but $x = a$. Here we also have to distinguish cases according to whether $y = n$ is standard or $y = b$:

$$(a \oplus n^*) = (a \oplus (n + 1)) = a = a^* = (a \oplus n)^*$$
$$(a \oplus a^*) = (a \oplus a) = a = a^* = (a \oplus a)^*$$

This is of course a bit more detailed than needed. For instance, since $a \oplus z = a$ whatever z is, we can immediately conclude $a \oplus a^* = a$. The remaining axioms can be verified the same way.

$\mathfrak{K}$ is thus a model of $\mathbf{Q}$. Its "addition" $\oplus$ is also commutative. But there are other sentences true in $\mathfrak{N}$ but false in $\mathfrak{K}$, and vice versa. For instance, $a \otimes a$, so $\mathfrak{K} \vDash \exists x\, x < x$ and $\mathfrak{K} \nvDash \forall x\, \neg x < x$. This shows that $\mathbf{Q} \nvdash \forall x\, \neg x < x$.

Problem mar.3. Prove that $\mathfrak{K}$ from Example mar.8 satisfies the remaining axioms of $\mathbf{Q}$,

$$\forall x\, (x \times \mathsf{o}) = \mathsf{o} \tag{Q_6}$$
$$\forall x\, \forall y\, (x \times y') = ((x \times y) + x) \tag{Q_7}$$
$$\forall x\, \forall y\, (x < y \leftrightarrow \exists z\, (z' + x) = y) \tag{Q_8}$$

Find a sentence only involving $\prime$ true in $\mathfrak{N}$ but false in $\mathfrak{K}$.

mod:mar:mdq: ex:model-L-of-Q **Example mar.9.** Consider the structure $\mathfrak{L}$ with domain $|\mathfrak{L}| = \mathbb{N} \cup \{a, b\}$ and interpretations $\prime^{\mathfrak{L}} = *$, $+^{\mathfrak{L}} = \oplus$ given by

x	x^*
n	$n + 1$
a	a
b	b

$x \oplus y$	m	a	b
n	$n + m$	b	a
a	a	b	a
b	b	b	a

Since $*$ is injective, 0 is not in its range, and every $x \in |\mathfrak{L}|$ other than 0 is, axioms Q_1–Q_3 are true in $\mathfrak{L}$. For any x, $x \oplus 0 = x$, so Q_4 is true as well. For Q_5, consider $x \oplus y^*$ and $(x \oplus y)^*$. They are equal if x and y are both standard, since then $*$ and $\oplus$ agree with $\prime$ and $+$. If x is non-standard, and y is standard, we have $x \oplus y^* = x = x^* = (x \oplus y)^*$. If x and y are both non-standard, we have four cases:

$$a \oplus a^* = b = b^* = (a \oplus a)^*$$
$$b \oplus b^* = a = a^* = (b \oplus b)^*$$
$$b \oplus a^* = b = b^* = (b \oplus y)^*$$
$$a \oplus b^* = a = a^* = (a \oplus b)^*$$

If x is standard, but y is non-standard, we have

$$n \oplus a^* = n \oplus a = b = b^* = (n \oplus a)^*$$
$$n \oplus b^* = n \oplus b = a = a^* = (n \oplus b)^*$$

So, $\mathfrak{L} \vDash Q_5$. However, $a \oplus 0 \neq 0 \oplus a$, so $\mathfrak{L} \nvDash \forall x\, \forall y\, (x + y) = (y + x)$.

Problem mar.4. Expand $\mathfrak{L}$ of Example mar.9 to include $\otimes$ and $\lessdot$ that interpret $\times$ and $<$. Show that your structure satisfies the remaining axioms of **Q**,

$$\forall x\, (x \times \mathsf{o}) = \mathsf{o} \tag{Q_6}$$
$$\forall x\, \forall y\, (x \times y') = ((x \times y) + x) \tag{Q_7}$$
$$\forall x\, \forall y\, (x < y \leftrightarrow \exists z\, (z' + x) = y) \tag{Q_8}$$

Problem mar.5. In $\mathfrak{L}$ of Example mar.9, $a^* = a$ and $b^* = b$. Is there a model of **Q** in which $a^* = b$ and $b^* = a$?

explanation We've explicitly constructed models of **Q** in which the non-standard elements live "beyond" the standard elements. In fact, that much is required by the axioms. A non-standard element x cannot be $\lessdot 0$, since $\mathbf{Q} \vdash \forall x\, \neg x < 0$ (see **??**). Also, for every n, $\mathbf{Q} \vdash \forall x\, (x < \overline{n}' \rightarrow (x = \overline{0} \lor x = \overline{1} \lor \dots \lor x = \overline{n}))$ (**??**), so we can't have $a \lessdot n$ for any $n > 0$.

mar.5 Models of PA

explanation Any non-standard model of **TA** is also one of **PA**. We know that non-standard models of **TA** and hence of **PA** exist. We also know that such non-standard models contain non-standard "numbers," i.e., elements of the domain that are "beyond" all the standard "numbers." But how are they arranged? How many are there? We've seen that models of the weaker theory **Q** can contain as few as a single non-standard number. But these simple structures are not models of **PA** or **TA**.

The key to understanding the structure of models of **PA** or **TA** is to see what facts are derivable in these theories. For instance, already **PA** proves that $\forall x\, x \neq x'$ and $\forall x\, \forall y\, (x + y) = (y + x)$, so this rules out simple structures (in which these sentences are false) as models of **PA**.

Suppose $\mathfrak{M}$ is a model of **PA**. Then if $\mathbf{PA} \vdash \varphi$, $\mathfrak{M} \vDash \varphi$. Let's again use $\mathbf{z}$ for $\mathsf{o}^{\mathfrak{M}}$, $*$ for $\prime^{\mathfrak{M}}$, $\oplus$ for $+^{\mathfrak{M}}$, $\otimes$ for $\times^{\mathfrak{M}}$, and $\lessdot$ for $<^{\mathfrak{M}}$. Any sentence φ then states some condition about $\mathbf{z}$, $*$, $\oplus$, $\otimes$, and $\lessdot$, and if $\mathfrak{M} \vDash \varphi$ that condition must be satisfied. For instance, if $\mathfrak{M} \vDash Q_1$, i.e., $\mathfrak{M} \vDash \forall x\, \forall y\, (x' = y' \rightarrow x = y)$, then $*$ must be injective.

Proposition mar.10. *In $\mathfrak{M}$, $\lessdot$ is a linear strict order, i.e., it satisfies:*

1. *Not $x \lessdot x$ for any $x \in |\mathfrak{M}|$.*

2. *If $x \lessdot y$ and $y \lessdot z$ then $x \lessdot z$.*

3. *For any $x \neq y$, $x \lessdot y$ or $y \lessdot x$*

Proof. **PA** proves:

1. $\forall x \, \neg x < x$

2. $\forall x \, \forall y \, \forall z \, ((x < y \land y < z) \to x < z)$

3. $\forall x \, \forall y \, ((x < y \lor y < x) \lor x = y))$ □

mod:mar:mpa: prop:M-discrete

Proposition mar.11. *$\mathbf{z}$ is the least element of $|\mathfrak{M}|$ in the $\lessdot$-ordering. For any x, $x \lessdot x^*$, and x^* is the $\lessdot$-least element with that property. For any x, there is a unique y such that $y^* = x$. (We call y the "predecessor" of x in $\mathfrak{M}$, and denote it by ${}^*x$.)*

Proof. Exercise. □

Problem mar.6. Find sentences in $\mathcal{L}_A$ derivable in **PA** (and hence true in $\mathfrak{N}$) which guarantee the properties of $\mathbf{z}$, $*$, and $\lessdot$ in Proposition mar.11

Proposition mar.12. *All standard elements of $\mathfrak{M}$ are less than (according to $\lessdot$) all non-standard elements.*

Proof. We'll use n as short for $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$, a standard element of $\mathfrak{M}$. Already **Q** proves that, for any $n \in \mathbb{N}$, $\forall x \, (x < \overline{n}' \to (x = \overline{0} \lor x = \overline{1} \lor \dots \lor x = \overline{n}))$. There are no elements that are $\lessdot \mathbf{z}$. So if n is standard and x is non-standard, we cannot have $x \lessdot n$. By definition, a non-standard element is one that isn't $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for any $n \in \mathbb{N}$, so $x \neq n$ as well. Since $\lessdot$ is a linear order, we must have $n \lessdot x$. □

Proposition mar.13. *Every nonstandard element x of $|\mathfrak{M}|$ is an element of the subset*

$$\dots {}^{***}x \lessdot {}^{**}x \lessdot {}^{*}x \lessdot x \lessdot x^* \lessdot x^{**} \lessdot x^{***} \lessdot \dots$$

We call this subset the block *of x and write it as $[x]$. It has no least and no greatest element. It can be characterized as the set of those $y \in |\mathfrak{M}|$ such that, for some standard n, $x \oplus n = y$ or $y \oplus n = x$.*

Proof. Clearly, such a set $[x]$ always exists since every element y of $|\mathfrak{M}|$ has a unique successor y^* and unique predecessor ${}^*y$. For successive elements y, y^* we have $y \lessdot y^*$ and y^* is the $\lessdot$-least element of $|\mathfrak{M}|$ such that y is $\lessdot$-less than it. Since always ${}^*y \lessdot y$ and $y \lessdot y^*$, $[x]$ has no least or greatest element. If $y \in [x]$ then $x \in [y]$, for then either $y^{*\dots*} = x$ or $x^{*\dots*} = y$. If $y^{*\dots*} = x$ (with n $*$'s), then $y \oplus n = x$ and conversely, since $\mathbf{PA} \vdash \forall x \, x'^{\dots\prime} = (x + \overline{n})$ (if n is the number of $\prime$'s). □

Proposition mar.14. *If $[x] \neq [y]$ and $x \lessdot y$, then for any $u \in [x]$ and any $v \in [y]$, $u \lessdot v$.*

Proof. Note that $\mathbf{PA} \vdash \forall x \, \forall y \, (x < y \to (x' < y \lor x' = y))$. Thus, if $u \lessdot v$, we also have $u \oplus n^* \lessdot v$ for any n if $[u] \neq [v]$.

Any $u \in [x]$ is $\lessdot y$: $x \lessdot y$ by assumption. If $u \lessdot x$, $u \lessdot y$ by transitivity. And if $x \lessdot u$ but $u \in [x]$, we have $u = x \oplus n^*$ for some n, and so $u \lessdot y$ by the fact just proved.

Now suppose that $v \in [y]$ is $\lessdot y$, i.e., $v \oplus m^* = y$ for some standard m. This rules out $v \lessdot x$, otherwise $y = v \oplus m^* \lessdot x$. Clearly also, $x \neq v$, otherwise $x \oplus m^* = v \oplus m^* = y$ and we would have $[x] = [y]$. So, $x \lessdot v$. But then also $x \oplus n^* \lessdot v$ for any n. Hence, if $x \lessdot u$ and $u \in [x]$, we have $u \lessdot v$. If $u \lessdot x$ then $u \lessdot v$ by transitivity.

Lastly, if $y \lessdot v$, $u \lessdot v$ since, as we've shown, $u \lessdot y$ and $y \lessdot v$. □

Corollary mar.15. *If $[x] \neq [y]$, $[x] \cap [y] = \emptyset$.*

Proof. Suppose $z \in [x]$ and $x \lessdot y$. Then $z \lessdot u$ for all $u \in [y]$. If $z \in [y]$, we would have $z \lessdot z$. Similarly if $y \lessdot x$. □

explanation

This means that the blocks themselves can be ordered in a way that respects $\lessdot$: $[x] \lessdot [y]$ iff $x \lessdot y$, or, equivalently, if $u \lessdot v$ for any $u \in [x]$ and $v \in [y]$. Clearly, the standard block $[0]$ is the least block. It intersects with no non-standard block, and no two non-standard blocks intersect either. Specifically, you cannot "reach" a different block by taking repeated successors or predecessors.

Proposition mar.16. *If x and y are non-standard, then $x \lessdot x \oplus y$ and $x \oplus y \notin [x]$.*

Proof. If y is nonstandard, then $y \neq \mathbf{z}$. $\mathbf{PA} \vdash \forall x \, (y \neq \mathsf{o} \to x < (x + y))$. Now suppose $x \oplus y \in [x]$. Since $x \lessdot x \oplus y$, we would have $x \oplus n^* = x \oplus y$. But $\mathbf{PA} \vdash \forall x \, \forall y \, \forall z \, ((x + y) = (x + z) \to y = z)$ (the cancellation law for addition). This would mean $y = n^*$ for some standard n; but y is assumed to be non-standard. □

Proposition mar.17. *There is no least non-standard block.*

Proof. $\mathbf{PA} \vdash \forall x \, \exists y \, ((y + y) = x \lor (y + y)' = x)$, i.e., that every x is divisible by 2 (possibly with remainder 1). If x is non-standard, so is y. By the preceding proposition, $y \lessdot y \oplus y$ and $y \oplus y \notin [y]$. Then also $y \lessdot (y \oplus y)^*$ and $(y \oplus y)^* \notin [y]$. But $x = y \oplus y$ or $x = (y \oplus y)^*$, so $y \lessdot x$ and $y \notin [x]$. □

Proposition mar.18. *There is no largest block.*

Proof. Exercise. □

Problem mar.7. Show that in a non-standard model of **PA**, there is no largest block.

mod:mar:mpa: prop:blocks-dense **Proposition mar.19.** *The ordering of the blocks is dense. That is, if $x \lessdot y$ and $[x] \neq [y]$, then there is a block $[z]$ distinct from both that is between them.*

Proof. Suppose $x \lessdot y$. As before, $x \oplus y$ is divisible by two (possibly with remainder): there is a $z \in |\mathfrak{M}|$ such that either $x \oplus y = z \oplus z$ or $x \oplus y = (z \oplus z)^*$. The element z is the "average" of x and y, and $x \lessdot z$ and $z \lessdot y$. □

Problem mar.8. Write out a detailed proof of Proposition mar.19. Which sentence must **PA** derive in order to guarantee the existence of z? Why is $x \lessdot z$ and $z \lessdot y$, and why is $[x] \neq [z]$ and $[z] \neq [y]$?

The non-standard blocks are therefore ordered like the rationals: they form a denumerable dense linear ordering without endpoints. One can show that any two such denumerable orderings are isomorphic. It follows that for any two enumerable non-standard models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of true arithmetic, their reducts to the language containing $<$ and $=$ only are isomorphic. Indeed, an isomorphism h can be defined as follows: the standard parts of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic to the standard model $\mathfrak{N}$ and hence to each other. The blocks making up the non-standard part are themselves ordered like the rationals and therefore isomorphic; an isomorphism of the blocks can be extended to an isomorphism *within* the blocks by matching up arbitrary elements in each, and then taking the image of the successor of x in $\mathfrak{M}_1$ to be the successor of the image of x in $\mathfrak{M}_2$. Note that it does *not* follow that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic in the full language of arithmetic (indeed, isomorphism is always relative to a language), as there are non-isomorphic ways to define addition and multiplication over $|\mathfrak{M}_1|$ and $|\mathfrak{M}_2|$. (This also follows from a famous theorem due to Vaught that the number of countable models of a complete theory cannot be 2.) explanation

mar.6 Computable Models of Arithmetic

The standard model $\mathfrak{N}$ has two nice features. Its domain is the natural numbers $\mathbb{N}$, i.e., its elements are just the kinds of things we want to talk about using the language of arithmetic, and the standard numeral $\overline{n}$ actually picks out n. The other nice feature is that the interpretations of the non-logical symbols of $\mathcal{L}_A$ are all *computable*. The successor, addition, and multiplication functions which serve as $\prime^{\mathfrak{N}}$, $+^{\mathfrak{N}}$, and $\times^{\mathfrak{N}}$ are computable functions of numbers. (Computable by Turing machines, or definable by primitive recursion, say.) And the less-than relation on $\mathfrak{N}$, i.e., $<^{\mathfrak{N}}$, is decidable. explanation

Non-standard models of arithmetical theories such as **Q** and **PA** must contain non-standard elements. Thus their domains typically include elements in addition to $\mathbb{N}$. However, any countable structure can be built on any denumerable set, including $\mathbb{N}$. So there are also non-standard models with domain $\mathbb{N}$. In such models $\mathfrak{M}$, of course, at least some numbers cannot play the roles they usually play, since some k must be different from $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for all $n \in \mathbb{N}$.

Definition mar.20. A structure $\mathfrak{M}$ for $\mathcal{L}_A$ is *computable* iff $|\mathfrak{M}| = \mathbb{N}$ and $\prime^{\mathfrak{M}}$, $+^{\mathfrak{M}}$, $\times^{\mathfrak{M}}$ are computable functions and $<^{\mathfrak{M}}$ is a decidable relation.

Example mar.21. Recall the structure $\mathfrak{K}$ from Example mar.8. Its domain was $|\mathfrak{K}| = \mathbb{N} \cup \{a\}$ and interpretations mod:mar:cmp: ex:comp-model-q

$$o^{\mathfrak{K}} = 0$$

$$\prime^{\mathfrak{K}}(x) = \begin{cases} x+1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases}$$

$$+^{\mathfrak{K}}(x,y) = \begin{cases} x+y & \text{if } x,\, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases}$$

$$\times^{\mathfrak{K}}(x,y) = \begin{cases} xy & \text{if } x,\, y \in \mathbb{N} \\ 0 & \text{if } x = 0 \text{ or } y = 0 \\ a & \text{otherwise} \end{cases}$$

$$<^{\mathfrak{K}} = \{\langle x,y\rangle : x,y \in \mathbb{N} \text{ and } x < y\} \cup \{\langle x,a\rangle : n \in |\mathfrak{K}|\}$$

But $|\mathfrak{K}|$ is denumerable and so is equinumerous with $\mathbb{N}$. For instance, $g\colon \mathbb{N} \to |\mathfrak{K}|$ with $g(0) = a$ and $g(n) = n+1$ for $n > 0$ is a bijection. We can turn it into an isomorphism between a new model $\mathfrak{K}'$ of **Q** and $\mathfrak{K}$. In $\mathfrak{K}'$, we have to assign different functions and relations to the symbols of $\mathcal{L}_A$, since different elements of $\mathbb{N}$ play the roles of standard and non-standard numbers.

Specifically, 0 now plays the role of a, not of the smallest standard number. The smallest standard number is now 1. So we assign $o^{\mathfrak{K}'} = 1$. The successor function is also different now: given a standard number, i.e., an $n > 0$, it still returns $n+1$. But 0 now plays the role of a, which is its own successor. So $\prime^{\mathfrak{K}'}(0) = 0$. For addition and multiplication we likewise have

$$+^{\mathfrak{K}'}(x,y) = \begin{cases} x+y-1 & \text{if } x,\, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\times^{\mathfrak{K}'}(x,y) = \begin{cases} 1 & \text{if } x = 1 \text{ or } y = 1 \\ xy - x - y + 2 & \text{if } x,\, y > 1 \\ 0 & \text{otherwise} \end{cases}$$

And we have $\langle x,y\rangle \in <^{\mathfrak{K}'}$ iff $x < y$ and $x > 0$ and $y > 0$, or if $y = 0$.

All of these functions are computable functions of natural numbers and $<^{\mathfrak{K}'}$ is a decidable relation on $\mathbb{N}$—but they are not the same functions as successor, addition, and multiplication on $\mathbb{N}$, and $<^{\mathfrak{K}'}$ is not the same relation as $<$ on $\mathbb{N}$.

Problem mar.9. Give a structure $\mathfrak{L}'$ with $|\mathfrak{L}'| = \mathbb{N}$ isomorphic to $\mathfrak{L}$ of Example mar.9.

explanation Example mar.21 shows that **Q** has computable non-standard models with domain $\mathbb{N}$. However, the following result shows that this is not true for models of **PA** (and thus also for models of **TA**).

Theorem mar.22 (Tennenbaum's Theorem). $\mathfrak{N}$ *is the only computable model of* **PA**.

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Bibliography