

## mar.1 Introduction

The *standard model* of arithmetic is the **structure**  $\mathfrak{N}$  with  $|\mathfrak{N}| = \mathbb{N}$  in which  $o$ ,  $l$ ,  $+$ ,  $\times$ , and  $<$  are interpreted as you would expect. That is,  $o$  is 0,  $l$  is the successor function,  $+$  is interpreted as addition and  $\times$  as multiplication of the numbers in  $\mathbb{N}$ . Specifically,

$$\begin{aligned} o^{\mathfrak{N}} &= 0 \\ l^{\mathfrak{N}}(n) &= n + 1 \\ +^{\mathfrak{N}}(n, m) &= n + m \\ \times^{\mathfrak{N}}(n, m) &= nm \end{aligned}$$

Of course, there are structures for  $\mathcal{L}_A$  that have domains other than  $\mathbb{N}$ . For instance, we can take  $\mathfrak{M}$  with domain  $|\mathfrak{M}| = \{a\}^*$  (the finite sequences of the single symbol  $a$ , i.e.,  $\emptyset, a, aa, aaa, \dots$ ), and interpretations

$$\begin{aligned} o^{\mathfrak{M}} &= \emptyset \\ l^{\mathfrak{M}}(s) &= s \frown a \\ +^{\mathfrak{M}}(n, m) &= a^{n+m} \\ \times^{\mathfrak{M}}(n, m) &= a^{nm} \end{aligned}$$

These two structures are “essentially the same” in the sense that the only difference is the **elements** of the **domains** but not how the **elements** of the **domains** are related among each other by the interpretation functions. We say that the two **structures** are *isomorphic*.

It is an easy consequence of the compactness theorem that any theory true in  $\mathfrak{N}$  also has models that are not isomorphic to  $\mathfrak{N}$ . Such structures are called *non-standard*. The interesting thing about them is that while the **elements** of a standard model (i.e.,  $\mathfrak{N}$ , but also all **structures** isomorphic to it) are exhausted by the values of the standard numerals  $\bar{n}$ , i.e.,

$$|\mathfrak{N}| = \{\text{Val}^{\mathfrak{N}}(\bar{n}) : n \in \mathbb{N}\}$$

that isn’t the case in non-standard models: if  $\mathfrak{M}$  is non-standard, then there is at least one  $x \in |\mathfrak{M}|$  such that  $x \neq \text{Val}^{\mathfrak{M}}(\bar{n})$  for all  $n$ .

These non-standard elements are pretty neat: they are “infinite natural numbers.” But their existence also explains, in a sense, the incompleteness phenomena. Consider an example, e.g., the consistency statement for Peano arithmetic,  $\text{Con}_{\mathbf{PA}}$ , i.e.,  $\neg \exists x \text{Prf}_{\mathbf{PA}}(x, \ulcorner \perp \urcorner)$ . Since  $\mathbf{PA}$  neither proves  $\text{Con}_{\mathbf{PA}}$  nor  $\neg \text{Con}_{\mathbf{PA}}$ , either can be consistently added to  $\mathbf{PA}$ . Since  $\mathbf{PA}$  is consistent,  $\mathfrak{N} \models \text{Con}_{\mathbf{PA}}$ , and consequently  $\mathfrak{N} \not\models \neg \text{Con}_{\mathbf{PA}}$ . So  $\mathfrak{N}$  is *not* a model of  $\mathbf{PA} \cup \{\neg \text{Con}_{\mathbf{PA}}\}$ , and all its models must be nonstandard. Models of  $\mathbf{PA} \cup \{\neg \text{Con}_{\mathbf{PA}}\}$  must contain some **element** that serves as the witness that makes  $\exists x \text{Prf}_{\mathbf{PA}}(\ulcorner \perp \urcorner)$  true, i.e., a Gödel number of a **derivation** of a contradiction from  $\mathbf{PA}$ . Such an **element** can’t be standard—since  $\mathbf{PA} \vdash \neg \text{Prf}_{\mathbf{PA}}(\bar{n}, \ulcorner \perp \urcorner)$  for every  $n$ .

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**Bibliography**