

## mar.1 Computable Models of Arithmetic

The standard model  $\mathfrak{N}$  has two nice features. Its domain is the natural numbers  $\mathbb{N}$ , i.e., its elements are just the kinds of things we want to talk about using the language of arithmetic, and the standard numeral  $\bar{n}$  actually picks out  $n$ . The other nice feature is that the interpretations of the non-logical symbols of  $\mathcal{L}_A$  are all *computable*. The successor, addition, and multiplication functions which serve as  $r^{\mathfrak{N}}$ ,  $+^{\mathfrak{N}}$ , and  $\times^{\mathfrak{N}}$  are computable functions of numbers. (Computable by Turing machines, or definable by primitive recursion, say.) And the less-than relation on  $\mathfrak{N}$ , i.e.,  $<^{\mathfrak{N}}$ , is decidable.

Non-standard models of arithmetical theories such as **Q** and **PA** must contain non-standard elements. Thus their domains typically include *elements* in addition to  $\mathbb{N}$ . However, any countable *structure* can be built on any *denumerable* set, including  $\mathbb{N}$ . So there are also non-standard models with domain  $\mathbb{N}$ . In such models  $\mathfrak{M}$ , of course, at least some numbers cannot play the roles they usually play, since some  $k$  must be different from  $\text{Val}^{\mathfrak{M}}(\bar{n})$  for all  $n \in \mathbb{N}$ .

**Definition mar.1.** A *structure*  $\mathfrak{M}$  for  $\mathcal{L}_A$  is *computable* iff  $|\mathfrak{M}| = \mathbb{N}$  and  $r^{\mathfrak{M}}$ ,  $+^{\mathfrak{M}}$ ,  $\times^{\mathfrak{M}}$  are computable functions and  $<^{\mathfrak{M}}$  is a decidable relation.

mod:mar:cmp:  
ex:comp-model-q

**Example mar.2.** Recall the structure  $\mathfrak{K}$  from ???. Its domain was  $|\mathfrak{K}| = \mathbb{N} \cup \{a\}$  and interpretations

$$\begin{aligned} o^{\mathfrak{K}} &= 0 \\ r^{\mathfrak{K}}(x) &= \begin{cases} x + 1 & \text{if } x \in \mathbb{N} \\ a & \text{if } x = a \end{cases} \\ +^{\mathfrak{K}}(x, y) &= \begin{cases} x + y & \text{if } x, y \in \mathbb{N} \\ a & \text{otherwise} \end{cases} \\ \times^{\mathfrak{K}}(x, y) &= \begin{cases} xy & \text{if } x, y \in \mathbb{N} \\ 0 & \text{if } x = 0 \text{ or } y = 0 \\ a & \text{otherwise} \end{cases} \\ <^{\mathfrak{K}} &= \{\langle x, y \rangle : x, y \in \mathbb{N} \text{ and } x < y\} \cup \{\langle x, a \rangle : n \in |\mathfrak{K}|\} \end{aligned}$$

But  $|\mathfrak{K}|$  is *denumerable* and so is equinumerous with  $\mathbb{N}$ . For instance,  $g: \mathbb{N} \rightarrow |\mathfrak{K}|$  with  $g(0) = a$  and  $g(n) = n + 1$  for  $n > 0$  is a *bijection*. We can turn it into an isomorphism between a new model  $\mathfrak{K}'$  of **Q** and  $\mathfrak{K}$ . In  $\mathfrak{K}'$ , we have to assign different functions and relations to the symbols of  $\mathcal{L}_A$ , since different *elements* of  $\mathbb{N}$  play the roles of standard and non-standard numbers.

Specifically, 0 now plays the role of  $a$ , not of the smallest standard number. The smallest standard number is now 1. So we assign  $o^{\mathfrak{K}'} = 1$ . The successor function is also different now: given a standard number, i.e., an  $n > 0$ , it still returns  $n + 1$ . But 0 now plays the role of  $a$ , which is its own successor. So

$\uparrow^{\mathfrak{R}'}(0) = 0$ . For addition and multiplication we likewise have

$$+^{\mathfrak{R}'}(x, y) = \begin{cases} x + y - 1 & \text{if } x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\times^{\mathfrak{R}'}(x, y) = \begin{cases} 1 & \text{if } x = 1 \text{ or } y = 1 \\ xy - x - y + 2 & \text{if } x, y > 1 \\ 0 & \text{otherwise} \end{cases}$$

And we have  $\langle x, y \rangle \in <^{\mathfrak{R}'}$  iff  $x < y$  and  $x > 0$  and  $y > 0$ , or if  $y = 0$ .

All of these functions are computable functions of natural numbers and  $<^{\mathfrak{R}'}$  is a decidable relation on  $\mathbb{N}$ —but they are not the same functions as successor, addition, and multiplication on  $\mathbb{N}$ , and  $<^{\mathfrak{R}'}$  is not the same relation as  $<$  on  $\mathbb{N}$ .

**Problem mar.1.** Give a structure  $\mathfrak{L}'$  with  $|\mathfrak{L}'| = \mathbb{N}$  isomorphic to  $\mathfrak{L}$  of ??.

explanation **Example mar.2** shows that  $\mathbf{Q}$  has computable non-standard models with domain  $\mathbb{N}$ . However, the following result shows that this is not true for models of  $\mathbf{PA}$  (and thus also for models of  $\mathbf{TA}$ ).

**Theorem mar.3 (Tennenbaum's Theorem).**  *$\mathfrak{N}$  is the only computable model of  $\mathbf{PA}$ .*

## Photo Credits

## Bibliography