lin.1 Compactness and Löwenheim–Skolem Properties

mod:lin:lsp: sec

We now give the obvious extensions of compactness and Löwenheim–Skolem to the case of abstract logics.

Definition lin.1. An abstract logic $\langle L, \models_L \rangle$ has the *Compactness Property* if each set Γ of $L(\mathcal{L})$-sentences is satisfiable whenever each finite $\Gamma_0 \subseteq \Gamma$ is satisfiable.

Definition lin.2. $\langle L, \models_L \rangle$ has the *Downward Löwenheim–Skolem property* if any satisfiable Γ has an enumerable model.

The notion of partial isomorphism from **??** is purely "algebraic" (i.e., given without reference to the sentences of the language but only to the constants provided by the language $\mathcal{L}$ of the structures), and hence it applies to the case of abstract logics. In case of first-order logic, we know from **??** that if two structures are partially isomorphic then they are elementarily equivalent. That proof does not carry over to abstract logics, for induction on formulas need not be available for arbitrary $\alpha \in L(\mathcal{L})$, but the theorem is true nonetheless, provided the Löwenheim–Skolem property holds.

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Theorem lin.3. *Suppose $\langle L, \models_L \rangle$ is a normal logic with the Löwenheim–Skolem property. Then any two structures that are partially isomorphic are elementarily equivalent in $\langle L, \models_L \rangle$.*

Proof. Suppose $\mathfrak{M} \simeq_p \mathfrak{N}$, but for some α also $\mathfrak{M} \models_L \alpha$ while $\mathfrak{N} \not\models_L \alpha$. By the Isomorphism Property we can assume that $|\mathfrak{M}|$ and $|\mathfrak{N}|$ are disjoint, and by the Expansion Property we can assume that $\alpha \in L(\mathcal{L})$ for a finite language $\mathcal{L}$. Let $\mathcal{I}$ be a set of partial isomorphisms between $\mathfrak{M}$ and $\mathfrak{N}$, and with no loss of generality also assume that if $p \in \mathcal{I}$ and $q \subseteq p$ then also $q \in \mathcal{I}$.

$|\mathfrak{M}|^{<\omega}$ is the set of finite sequences of elements of $|\mathfrak{M}|$. Let S be the ternary relation over $|\mathfrak{M}|^{<\omega}$ representing concatenation, i.e., if $\mathbf{a}, \mathbf{b}, \mathbf{c} \in |\mathfrak{M}|^{<\omega}$ then $S(\mathbf{a}, \mathbf{b}, \mathbf{c})$ holds if and only if $\mathbf{c}$ is the concatenation of $\mathbf{a}$ and $\mathbf{b}$; and let T be the ternary relation such that $T(\mathbf{a}, b, \mathbf{c})$ holds for $b \in M$ and $\mathbf{a}, \mathbf{c} \in |\mathfrak{M}|^{<\omega}$ if and only if $\mathbf{a} = a_1, \dots a_n$ and $\mathbf{c} = a_1, \dots a_n, b$. Pick new 3-place predicate symbols P and Q and form the structure $\mathfrak{M}^*$ having the universe $|\mathfrak{M}| \cup |\mathfrak{M}|^{<\omega}$, having $\mathfrak{M}$ as a substructure, and interpreting P and Q by the concatenation relations S and T (so $\mathfrak{M}^*$ is in the language $\mathcal{L} \cup \{P, Q\}$).

Define $|\mathfrak{N}|^{<\omega}$, S', T', P', Q' and $\mathfrak{N}^*$ analogously. Since by hypothesis $\mathfrak{M} \simeq_p \mathfrak{N}$, there is a relation I between $|\mathfrak{M}|^{<\omega}$ and $|\mathfrak{N}|^{<\omega}$ such that $I(\mathbf{a}, \mathbf{b})$ holds if and only if $\mathbf{a}$ and $\mathbf{b}$ are isomorphic and satisfy the back-and-forth condition of **??**. Now, let $\mathfrak{M}$ be the structure whose domain is the union of the domains of $\mathfrak{M}^*$ and $\mathfrak{N}^*$, having $\mathfrak{M}^*$ and $\mathfrak{N}^*$ as substructures, in the language with one extra binary predicate symbol R interpreted by the relation I and predicate symbols denoting the domains $|\mathfrak{M}|^*$ and $|\mathfrak{N}|*$.

The crucial observation is that in the language of the structure $\mathfrak{M}$ there is a *first-order* sentence θ_1 true in $\mathfrak{M}$ saying that $\mathfrak{M} \models_L \alpha$ and $\mathfrak{N} \not\models_L \alpha$

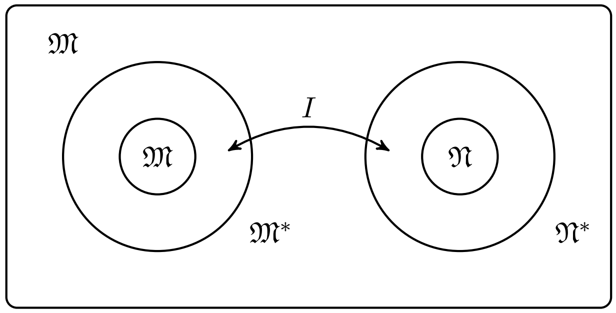

Figure 1: The structure $\mathfrak{M}$ with the internal partial isomorphism.

(this requires the Relativization Property), as well as a *first-order* sentence θ_2 true in $\mathfrak{M}$ saying that $\mathfrak{M} \simeq_p \mathfrak{N}$ via the partial isomorphism I. By the Löwenheim–Skolem Property, θ_1 and θ_2 are jointly true in an enumerable model $\mathfrak{M}_0$ containing partially isomorphic substructures $\mathfrak{M}_0$ and $\mathfrak{N}_0$ such that $\mathfrak{M}_0 \models_L \alpha$ and $\mathfrak{N}_0 \not\models_L \alpha$. But enumerable partially isomorphic structures are in fact isomorphic by **??**, contradicting the Isomorphism Property of normal abstract logics. $\square$

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Bibliography