

## Chapter udf

# Lindström's Theorem

### lin.1 Introduction

In this chapter we aim to prove Lindström's characterization of first-order logic as the maximal logic for which (given certain further constraints) the Compactness and the Downward Löwenheim–Skolem theorems hold (?? and ??). First, we need a more general characterization of the general class of logics to which the theorem applies. We will restrict ourselves to *relational* languages, i.e., languages which only contain **predicate symbols** and individual constants, but no **function symbols**.

### lin.2 Abstract Logics

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**Definition lin.1.** An *abstract logic* is a pair  $\langle L, \models_L \rangle$ , where  $L$  is a function that assigns to each **language**  $\mathcal{L}$  a set  $L(\mathcal{L})$  of **sentences**, and  $\models_L$  is a relation between **structures** for the **language**  $\mathcal{L}$  and **elements** of  $L(\mathcal{L})$ . In particular,  $\langle F, \models \rangle$  is ordinary first-order logic, i.e.,  $F$  is the function assigning to the **language**  $\mathcal{L}$  the set of first-order **sentences** built from the constants in  $\mathcal{L}$ , and  $\models$  is the satisfaction relation of first-order logic.

Notice that we are still employing the same notion of **structure** for a given **language** as for first-order logic, but we do not presuppose that **sentences** are built up from the basic symbols in  $\mathcal{L}$  in the usual way, nor that the relation  $\models_L$  is recursively defined in the same way as for first-order logic. So for instance the definition, being completely general, is intended to capture the case where **sentences** in  $\langle L, \models_L \rangle$  contain infinitely long conjunctions or disjunction, or quantifiers other than  $\exists$  and  $\forall$  (e.g., “there are infinitely many  $x$  such that ...”), or perhaps infinitely long quantifier prefixes. To emphasize that “**sentences**” in  $L(\mathcal{L})$  need not be ordinary **sentences** of first-order logic, in this chapter we use **variables**  $\alpha, \beta, \dots$  to range over them, and reserve  $\varphi, \psi, \dots$  for ordinary first-order **formulas**.

**Definition lin.2.** Let  $\text{Mod}_L(\alpha)$  denote the class  $\{\mathfrak{M} : \mathfrak{M} \models_L \alpha\}$ . If the **language** needs to be made explicit, we write  $\text{Mod}_L^{\mathcal{L}}(\alpha)$ . Two **structures**  $\mathfrak{M}$  and  $\mathfrak{N}$  for  $\mathcal{L}$  are *elementarily equivalent in*  $\langle L, \models_L \rangle$ , written  $\mathfrak{M} \equiv_L \mathfrak{N}$ , if the same **sentences** from  $L(\mathcal{L})$  are true in each.

**Definition lin.3.** An abstract logic  $\langle L, \models_L \rangle$  for the **language**  $\mathcal{L}$  is *normal* if it satisfies the following properties:

1. (*L-Monotonicity*) For **languages**  $\mathcal{L}$  and  $\mathcal{L}'$ , if  $\mathcal{L} \subseteq \mathcal{L}'$ , then  $L(\mathcal{L}) \subseteq L(\mathcal{L}')$ .
2. (*Expansion Property*) For each  $\alpha \in L(\mathcal{L})$  there is a *finite* subset  $\mathcal{L}'$  of  $\mathcal{L}$  such that the relation  $\mathfrak{M} \models_L \alpha$  depends only on the reduct of  $\mathfrak{M}$  to  $\mathcal{L}'$ ; i.e., if  $\mathfrak{M}$  and  $\mathfrak{N}$  have the same reduct to  $\mathcal{L}'$  then  $\mathfrak{M} \models_L \alpha$  if and only if  $\mathfrak{N} \models_L \alpha$ .
3. (*Isomorphism Property*) If  $\mathfrak{M} \models_L \alpha$  and  $\mathfrak{M} \simeq \mathfrak{N}$  then also  $\mathfrak{N} \models_L \alpha$ .
4. (*Renaming Property*) The relation  $\models_L$  is preserved under renaming: if the **language**  $\mathcal{L}'$  is obtained from  $\mathcal{L}$  by replacing each symbol  $P$  by a symbol  $P'$  of the same arity and each constant  $c$  by a distinct constant  $c'$ , then for each **structure**  $\mathfrak{M}$  and **sentence**  $\alpha$ ,  $\mathfrak{M} \models_L \alpha$  if and only if  $\mathfrak{M}' \models_L \alpha'$ , where  $\mathfrak{M}'$  is the  $\mathcal{L}'$ -**structure** corresponding to  $\mathfrak{M}$  and  $\alpha' \in L(\mathcal{L}')$ .
5. (*Boolean Property*) The abstract logic  $\langle L, \models_L \rangle$  is closed under the Boolean connectives in the sense that for each  $\alpha \in L(\mathcal{L})$  there is a  $\beta \in L(\mathcal{L})$  such that  $\mathfrak{M} \models_L \beta$  if and only if  $\mathfrak{M} \not\models_L \alpha$ , and for each  $\alpha$  and  $\beta$  there is a  $\gamma$  such that  $\text{Mod}_L(\gamma) = \text{Mod}_L(\alpha) \cap \text{Mod}_L(\beta)$ . Similarly for atomic **formulas** and the other connectives.
6. (*Quantifier Property*) For each constant  $c$  in  $\mathcal{L}$  and  $\alpha \in L(\mathcal{L})$  there is a  $\beta \in L(\mathcal{L})$  such that

$$\text{Mod}_L^{\mathcal{L}'}(\beta) = \{\mathfrak{M} : (\mathfrak{M}, a) \in \text{Mod}_L^{\mathcal{L}}(\alpha) \text{ for some } a \in |\mathfrak{M}|\},$$

where  $\mathcal{L}' = \mathcal{L} \setminus \{c\}$  and  $(\mathfrak{M}, a)$  is the expansion of  $\mathfrak{M}$  to  $\mathcal{L}$  assigning  $a$  to  $c$ .

7. (*Relativization Property*) Given a **sentence**  $\alpha \in L(\mathcal{L})$  and symbols  $R, c_1, \dots, c_n$  not in  $\mathcal{L}$ , there is a **sentence**  $\beta \in L(\mathcal{L} \cup \{R, c_1, \dots, c_n\})$  called the *relativization* of  $\alpha$  to  $R(x, c_1, \dots, c_n)$ , such that for each **structure**  $\mathfrak{M}$ :

$$(\mathfrak{M}, X, b_1, \dots, b_n) \models_L \beta \text{ if and only if } \mathfrak{N} \models_L \alpha,$$

where  $\mathfrak{N}$  is the substructure of  $\mathfrak{M}$  with **domain**  $|\mathfrak{N}| = \{a \in |\mathfrak{M}| : R^{\mathfrak{M}}(a, b_1, \dots, b_n)\}$  (see ??), and  $(\mathfrak{M}, X, b_1, \dots, b_n)$  is the expansion of  $\mathfrak{M}$  interpreting  $R, c_1, \dots, c_n$  by  $X, b_1, \dots, b_n$ , respectively (with  $X \subseteq M^{n+1}$ ).

**Definition lin.4.** Given two abstract logics  $\langle L_1, \models_{L_1} \rangle$  and  $\langle L_2, \models_{L_2} \rangle$  we say that the latter is *at least as expressive* as the former, written  $\langle L_1, \models_{L_1} \rangle \leq \langle L_2, \models_{L_2} \rangle$ , if for each **language**  $\mathcal{L}$  and **sentence**  $\alpha \in L_1(\mathcal{L})$  there is a **sentence**  $\beta \in L_2(\mathcal{L})$  such that  $\text{Mod}_{L_1}^{\mathcal{L}}(\alpha) = \text{Mod}_{L_2}^{\mathcal{L}}(\beta)$ . The logics  $\langle L_1, \models_{L_1} \rangle$  and  $\langle L_2, \models_{L_2} \rangle$  are *equivalent* if  $\langle L_1, \models_{L_1} \rangle \leq \langle L_2, \models_{L_2} \rangle$  and  $\langle L_2, \models_{L_2} \rangle \leq \langle L_1, \models_{L_1} \rangle$ .

*Remark 1.* First-order logic, i.e., the abstract logic  $\langle F, \models \rangle$ , is normal. In fact, the above properties are mostly straightforward for first-order logic. We just remark that the expansion property comes down to extensionality, and that the relativization of a **sentence**  $\alpha$  to  $R(x, c_1, \dots, c_n)$  is obtained by replacing each **subformula**  $\forall x \beta$  by  $\forall x (R(x, c_1, \dots, c_n) \rightarrow \beta)$ . Moreover, if  $\langle L, \models_L \rangle$  is normal, then  $\langle F, \models \rangle \leq \langle L, \models_L \rangle$ , as can be shown by induction on first-order **formulas**. Accordingly, with no loss in generality, we can assume that every first-order **sentence** belongs to every normal logic.

### lin.3 Compactness and Löwenheim–Skolem Properties

mod:lin:lsp: sec We now give the obvious extensions of compactness and Löwenheim–Skolem to the case of abstract logics.

**Definition lin.5.** An abstract logic  $\langle L, \models_L \rangle$  has the *Compactness Property* if each set  $\Gamma$  of  $L(\mathcal{L})$ -**sentences** is satisfiable whenever each finite  $\Gamma_0 \subseteq \Gamma$  is satisfiable.

**Definition lin.6.**  $\langle L, \models_L \rangle$  has the *Downward Löwenheim–Skolem property* if any satisfiable  $\Gamma$  has an **enumerable** model.

The notion of partial isomorphism from ?? is purely “algebraic” (i.e., given without reference to the **sentences** of the language but only to the constants provided by the **language**  $\mathcal{L}$  of the **structures**), and hence it applies to the case of abstract logics. In case of first-order logic, we know from ?? that if two **structures** are partially isomorphic then they are elementarily equivalent. That proof does not carry over to abstract logics, for induction on **formulas** need not be available for arbitrary  $\alpha \in L(\mathcal{L})$ , but the theorem is true nonetheless, provided the Löwenheim–Skolem property holds.

mod:lin:lsp: thm:abstract-p-isom **Theorem lin.7.** *Suppose  $\langle L, \models_L \rangle$  is a normal logic with the Löwenheim–Skolem property. Then any two **structures** that are partially isomorphic are elementarily equivalent in  $\langle L, \models_L \rangle$ .*

*Proof.* Suppose  $\mathfrak{M} \simeq_p \mathfrak{N}$ , but for some  $\alpha$  also  $\mathfrak{M} \models_L \alpha$  while  $\mathfrak{N} \not\models_L \alpha$ . By the Isomorphism Property we can assume that  $|\mathfrak{M}|$  and  $|\mathfrak{N}|$  are disjoint, and by the Expansion Property we can assume that  $\alpha \in L(\mathcal{L})$  for a finite **language**  $\mathcal{L}$ . Let  $\mathcal{I}$  be a set of partial isomorphisms between  $\mathfrak{M}$  and  $\mathfrak{N}$ , and with no loss of generality also assume that if  $p \in \mathcal{I}$  and  $q \subseteq p$  then also  $q \in \mathcal{I}$ .

$|\mathfrak{M}|^{<\omega}$  is the set of finite sequences of **elements** of  $|\mathfrak{M}|$ . Let  $S$  be the ternary relation over  $|\mathfrak{M}|^{<\omega}$  representing concatenation, i.e., if  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in |\mathfrak{M}|^{<\omega}$  then

$S(\mathbf{a}, \mathbf{b}, \mathbf{c})$  holds if and only if  $\mathbf{c}$  is the concatenation of  $\mathbf{a}$  and  $\mathbf{b}$ ; and let  $T$  be the ternary relation such that  $T(\mathbf{a}, b, \mathbf{c})$  holds for  $b \in M$  and  $\mathbf{a}, \mathbf{c} \in |\mathfrak{M}|^{<\omega}$  if and only if  $\mathbf{a} = a_1, \dots, a_n$  and  $\mathbf{c} = a_1, \dots, a_n, b$ . Pick new 3-place predicate symbols  $P$  and  $Q$  and form the structure  $\mathfrak{M}^*$  having the universe  $|\mathfrak{M}| \cup |\mathfrak{M}|^{<\omega}$ , having  $\mathfrak{M}$  as a substructure, and interpreting  $P$  and  $Q$  by the concatenation relations  $S$  and  $T$  (so  $\mathfrak{M}^*$  is in the language  $\mathcal{L} \cup \{P, Q\}$ ).

Define  $|\mathfrak{N}|^{<\omega}$ ,  $S'$ ,  $T'$ ,  $P'$ ,  $Q'$  and  $\mathfrak{N}^*$  analogously. Since by hypothesis  $\mathfrak{M} \simeq_p \mathfrak{N}$ , there is a relation  $I$  between  $|\mathfrak{M}|^{<\omega}$  and  $|\mathfrak{N}|^{<\omega}$  such that  $I(\mathbf{a}, \mathbf{b})$  holds if and only if  $\mathbf{a}$  and  $\mathbf{b}$  are isomorphic and satisfy the back-and-forth condition of ???. Now, let  $\mathfrak{M}$  be the structure whose domain is the union of the domains of  $\mathfrak{M}^*$  and  $\mathfrak{N}^*$ , having  $\mathfrak{M}^*$  and  $\mathfrak{N}^*$  as substructures, in the language with one extra binary predicate symbol  $R$  interpreted by the relation  $I$  and predicate symbols denoting the domains  $|\mathfrak{M}^*|$  and  $|\mathfrak{N}^*|$ .

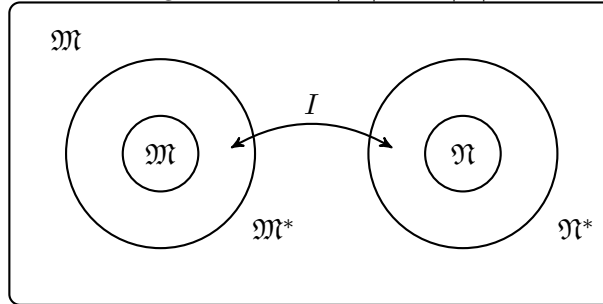


Figure lin.1: The structure  $\mathfrak{M}$  with the internal partial isomorphism.

The crucial observation is that in the language of the structure  $\mathfrak{M}$  there is a first-order sentence  $\theta_1$  true in  $\mathfrak{M}$  saying that  $\mathfrak{M} \models_L \alpha$  and  $\mathfrak{N} \not\models_L \alpha$  (this requires the Relativization Property), as well as a first-order sentence  $\theta_2$  true in  $\mathfrak{M}$  saying that  $\mathfrak{M} \simeq_p \mathfrak{N}$  via the partial isomorphism  $I$ . By the Löwenheim–Skolem Property,  $\theta_1$  and  $\theta_2$  are jointly true in an enumerable model  $\mathfrak{M}_0$  containing partially isomorphic substructures  $\mathfrak{M}_0$  and  $\mathfrak{N}_0$  such that  $\mathfrak{M}_0 \models_L \alpha$  and  $\mathfrak{N}_0 \not\models_L \alpha$ . But enumerable partially isomorphic structures are in fact isomorphic by ??, contradicting the Isomorphism Property of normal abstract logics.  $\square$

#### lin.4 Lindström’s Theorem

**Lemma lin.8.** *Suppose  $\alpha \in L(\mathcal{L})$ , with  $\mathcal{L}$  finite, and assume also that there is an  $n \in \mathbb{N}$  such that for any two structures  $\mathfrak{M}$  and  $\mathfrak{N}$ , if  $\mathfrak{M} \equiv_n \mathfrak{N}$  and  $\mathfrak{M} \models_L \alpha$  then also  $\mathfrak{N} \models_L \alpha$ . Then  $\alpha$  is equivalent to a first-order sentence, i.e., there is a first-order  $\theta$  such that  $\text{Mod}_L(\alpha) = \text{Mod}_L(\theta)$ .*

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*Proof.* Let  $n$  be such that any two  $n$ -equivalent structures  $\mathfrak{M}$  and  $\mathfrak{N}$  agree on the value assigned to  $\alpha$ . Recall ???: there are only finitely many first-order sentences in a finite language that have quantifier rank no greater than  $n$ , up to

logical equivalence. Now, for each fixed **structure**  $\mathfrak{M}$  let  $\theta_{\mathfrak{M}}$  be the conjunction of all first-order **sentences**  $\alpha$  true in  $\mathfrak{M}$  with  $\text{qr}(\alpha) \leq n$  (this conjunction is finite), so that  $\mathfrak{N} \models \theta_{\mathfrak{M}}$  if and only if  $\mathfrak{N} \equiv_n \mathfrak{M}$ . Then put  $\theta = \bigvee \{\theta_{\mathfrak{M}} : \mathfrak{M} \models_L \alpha\}$ ; this disjunction is also finite (up to logical equivalence).

The conclusion  $\text{Mod}_L(\alpha) = \text{Mod}_L(\theta)$  follows. In fact, if  $\mathfrak{N} \models_L \theta$  then for some  $\mathfrak{M} \models_L \alpha$  we have  $\mathfrak{N} \models \theta_{\mathfrak{M}}$ , whence also  $\mathfrak{N} \models_L \alpha$  (by the hypothesis of the lemma). Conversely, if  $\mathfrak{N} \models_L \alpha$  then  $\theta_{\mathfrak{N}}$  is a disjunct in  $\theta$ , and since  $\mathfrak{N} \models \theta_{\mathfrak{N}}$ , also  $\mathfrak{N} \models_L \theta$ .  $\square$

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**Theorem lin.9 (Lindström's Theorem).** *Suppose  $\langle L, \models_L \rangle$  has the Compactness and the Löwenheim–Skolem Properties. Then  $\langle L, \models_L \rangle \leq \langle F, \models \rangle$  (so  $\langle L, \models_L \rangle$  is equivalent to first-order logic).*

*Proof.* By **Lemma lin.8**, it suffices to show that for any  $\alpha \in L(\mathcal{L})$ , with  $\mathcal{L}$  finite, there is  $n \in \mathbb{N}$  such that for any two **structures**  $\mathfrak{M}$  and  $\mathfrak{N}$ : if  $\mathfrak{M} \equiv_n \mathfrak{N}$  then  $\mathfrak{M}$  and  $\mathfrak{N}$  agree on  $\alpha$ . For then  $\alpha$  is equivalent to a first-order **sentence**, from which  $\langle L, \models_L \rangle \leq \langle F, \models \rangle$  follows. Since we are working in a finite, purely relational **language**, by ?? we can replace the statement that  $\mathfrak{M} \equiv_n \mathfrak{N}$  by the corresponding algebraic statement that  $I_n(\emptyset, \emptyset)$ .

Given  $\alpha$ , suppose towards a contradiction that for each  $n$  there are **structures**  $\mathfrak{M}_n$  and  $\mathfrak{N}_n$  such that  $I_n(\emptyset, \emptyset)$ , but (say)  $\mathfrak{M}_n \models_L \alpha$  whereas  $\mathfrak{N}_n \not\models_L \alpha$ . By the Isomorphism Property we can assume that all the  $\mathfrak{M}_n$ 's interpret the constants of the language by the same objects; furthermore, since there are only finitely many atomic **sentences** in the language, we may also assume that they satisfy the same atomic **sentences** (we can take a subsequence of the  $\mathfrak{M}$ 's otherwise). Let  $\mathfrak{M}$  be the union of all the  $\mathfrak{M}_n$ 's, i.e., the unique minimal **structure** having each  $\mathfrak{M}_n$  as a substructure. As in the proof of **Theorem lin.7**, let  $\mathfrak{M}^*$  be the extension of  $\mathfrak{M}$  with **domain**  $|\mathfrak{M}| \cup |\mathfrak{M}|^{<\omega}$ , in the expanded **language** comprising the concatenation predicates  $P$  and  $Q$ .

Similarly, define  $\mathfrak{N}_n$ ,  $\mathfrak{N}$  and  $\mathfrak{N}^*$ . Now let  $\mathfrak{M}$  be the **structure** whose **domain** comprises the **domains** of  $\mathfrak{M}^*$  and  $\mathfrak{N}^*$  as well as the natural numbers  $\mathbb{N}$  along with their natural ordering  $\leq$ , in the **language** with extra predicates representing the **domains**  $|\mathfrak{M}|$ ,  $|\mathfrak{N}|$ ,  $|\mathfrak{M}|^{<\omega}$  and  $|\mathfrak{N}|^{<\omega}$  as well as predicates coding the domains of  $\mathfrak{M}_n$  and  $\mathfrak{N}_n$  in the sense that:

$$\begin{aligned} |\mathfrak{M}_n| &= \{a \in |\mathfrak{M}| : R(a, n)\}; & |\mathfrak{N}_n| &= \{a \in |\mathfrak{N}| : S(a, n)\}; \\ |\mathfrak{M}_n|^{<\omega} &= \{a \in |\mathfrak{M}|^{<\omega} : R(a, n)\}; & |\mathfrak{N}_n|^{<\omega} &= \{a \in |\mathfrak{N}|^{<\omega} : S(a, n)\}. \end{aligned}$$

The **structure**  $\mathfrak{M}$  also has a ternary relation  $J$  such that  $J(n, \mathbf{a}, \mathbf{b})$  holds if and only if  $I_n(\mathbf{a}, \mathbf{b})$ .

Now there is a **sentence**  $\theta$  in the **language**  $\mathcal{L}$  augmented by  $R, S, J$ , etc., saying that  $\leq$  is a discrete linear ordering with first but no last element and such that  $\mathfrak{M}_n \models \alpha$ ,  $\mathfrak{N}_n \not\models \alpha$ , and for each  $n$  in the ordering,  $J(n, \mathbf{a}, \mathbf{b})$  holds if and only if  $I_n(\mathbf{a}, \mathbf{b})$ .

Using the Compactness Property, we can find a model  $\mathfrak{M}^*$  of  $\theta$  in which the ordering contains a non-standard element  $n^*$ . In particular then  $\mathfrak{M}^*$  will

contain substructures  $\mathfrak{M}_{n^*}$  and  $\mathfrak{N}_{n^*}$  such that  $\mathfrak{M}_{n^*} \models_L \alpha$  and  $\mathfrak{N}_{n^*} \not\models_L \alpha$ . But now we can define a set  $\mathcal{I}$  of pairs of  $k$ -tuples from  $|\mathfrak{M}_{n^*}|$  and  $|\mathfrak{N}_{n^*}|$  by putting  $\langle \mathbf{a}, \mathbf{b} \rangle \in \mathcal{I}$  if and only if  $J(n^* - k, \mathbf{a}, \mathbf{b})$ , where  $k$  is the length of  $\mathbf{a}$  and  $\mathbf{b}$ . Since  $n^*$  is non-standard, for each standard  $k$  we have that  $n^* - k > 0$ , and the set  $\mathcal{I}$  witnesses the fact that  $\mathfrak{M}_{n^*} \simeq_p \mathfrak{N}_{n^*}$ . But by [Theorem lin.7](#),  $\mathfrak{M}_{n^*}$  is  $L$ -equivalent to  $\mathfrak{N}_{n^*}$ , a contradiction.  $\square$

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# Bibliography