lin.1 Abstract Logics

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Definition lin.1. An *abstract logic* is a pair $\langle L, \models_L \rangle$, where L is a function that assigns to each language $\mathcal{L}$ a set $L(\mathcal{L})$ of sentences, and $\models_L$ is a relation between structures for the language $\mathcal{L}$ and elements of $L(\mathcal{L})$. In particular, $\langle F, \models \rangle$ is ordinary first-order logic, i.e., F is the function assigning to the language $\mathcal{L}$ the set of first-order sentences built from the constants in $\mathcal{L}$, and $\models$ is the satisfaction relation of first-order logic.

Notice that we are still employing the same notion of structure for a given language as for first-order logic, but we do not presuppose that sentences are build up from the basic symbols in $\mathcal{L}$ in the usual way, nor that the relation $\models_L$ is recursively defined in the same way as for first-order logic. So for instance the definition, being completely general, is intended to capture the case where sentences in $\langle L, \models_L \rangle$ contain infinitely long conjunctions or disjunction, or quantifiers other than $\exists$ and $\forall$ (e.g., "there are infinitely many x such that ..."), or perhaps infinitely long quantifier prefixes. To emphasize that "sentences" in $L(\mathcal{L})$ need not be ordinary sentences of first-order logic, in this chapter we use variables α, β, ... to range over them, and reserve φ, ψ, ... for ordinary first-order formulas.

Definition lin.2. Let $\mathrm{Mod}_L(\alpha)$ denote the class $\{\mathfrak{M} : \mathfrak{M} \models_L \alpha\}$. If the language needs to be made explicit, we write $\mathrm{Mod}^{\mathcal{L}}_L(\alpha)$. Two structures $\mathfrak{M}$ and $\mathfrak{N}$ for $\mathcal{L}$ are *elementarily equivalent in* $\langle L, \models_L \rangle$, written $\mathfrak{M} \equiv_L \mathfrak{N}$, if the same sentences from $L(\mathcal{L})$ are true in each.

Definition lin.3. An abstract logic $\langle L, \models_L \rangle$ for the language $\mathcal{L}$ is *normal* if it satisfies the following properties:

1. (*L-Monotony*) For languages $\mathcal{L}$ and $\mathcal{L}'$, if $\mathcal{L} \subseteq \mathcal{L}'$, then $L(\mathcal{L}) \subseteq L(\mathcal{L}')$.

2. (*Expansion Property*) For each $\alpha \in L(\mathcal{L})$ there is a *finite* subset $\mathcal{L}'$ of $\mathcal{L}$ such that the relation $\mathfrak{M} \models_L \alpha$ depends only on the reduct of $\mathfrak{M}$ to $\mathcal{L}'$; i.e., if $\mathfrak{M}$ and $\mathfrak{N}$ have the same reduct to $\mathcal{L}'$ then $\mathfrak{M} \models_L \alpha$ if and only if $\mathfrak{N} \models_L \alpha$.

3. (*Isomorphism Property*) If $\mathfrak{M} \models_L \alpha$ and $\mathfrak{M} \simeq \mathfrak{N}$ then also $\mathfrak{N} \models_L \alpha$.

4. (*Renaming Property*) The relation $\models_L$ is preserved under renaming: if the language $\mathcal{L}'$ is obtained from $\mathcal{L}$ by replacing each symbol P by a symbol P' of the same arity and each constant c by a distinct constant c', then for each structure $\mathfrak{M}$ and sentence α, $\mathfrak{M} \models_L \alpha$ if and only if $\mathfrak{M}' \models_L \alpha'$, where $\mathfrak{M}'$ is the $\mathcal{L}'$-structure corresponding to $\mathcal{L}$ and $\alpha' \in L(\mathcal{L}')$.

5. (*Boolean Property*) The abstract logic $\langle L, \models_L \rangle$ is closed under the Boolean connectives in the sense that for each $\alpha \in L(\mathcal{L})$ there is a $\beta \in L(\mathcal{L})$ such that $\mathfrak{M} \models_L \beta$ if and only if $\mathfrak{M} \not\models_L \alpha$, and for each α and β there is a γ

such that $\mathrm{Mod}_L(\gamma) = \mathrm{Mod}_L(\alpha) \cap \mathrm{Mod}_L(\beta)$. Similarly for atomic formulas and the other connectives.

6. (*Quantifier Property*) For each constant c in $\mathcal{L}$ and $\alpha \in L(\mathcal{L})$ there is a $\beta \in L(\mathcal{L})$ such that

$$\mathrm{Mod}_L^{\mathcal{L}'}(\beta) = \{\mathfrak{M} : (\mathfrak{M}, a)\} \in \mathrm{Mod}_L^{\mathcal{L}}(\alpha) \text{ for some } a \in |\mathfrak{M}|\},$$

where $\mathcal{L}' = \mathcal{L} \setminus \{c\}$ and $(\mathfrak{M}, a)$ is the expansion of $\mathfrak{M}$ to $\mathcal{L}$ assigning a to c.

7. (*Relativization Property*) Given a sentence $\alpha \in L(\mathcal{L})$ and symbols R, c_1, $\dots$, c_n not in $\mathcal{L}$, there is a sentence $\beta \in L(\mathcal{L} \cup \{R, c_1, \dots, c_n\})$ called the *relativization* of α to $R(x, c_1, \dots c_n)$, such that for each structure $\mathfrak{M}$:

$$(\mathfrak{M}, X, b_1, \dots, b_n) \models_L \beta) \text{ if and only if } \mathfrak{N} \models_L \alpha,$$

where $\mathfrak{N}$ is the substructure of $\mathfrak{M}$ with domain $|\mathfrak{N}| = \{a \in |\mathfrak{M}| : R^{\mathfrak{M}}(a, b_1, \dots, b_n)\}$ (see **??**), and $(\mathfrak{M}, X, b_1, \dots, b_n)$ is the expansion of $\mathfrak{M}$ interpreting R, c_1, $\dots$, c_n by X, b_1, $\dots$, b_n, respectively (with $X \subseteq M^{n+1}$).

Definition lin.4. Given two abstract logics $\langle L_1, \models_{L_1}\rangle$ and $\langle L_2, \models_{L_2}\rangle$ we say that the latter is *at least as expressive* as the former, written $\langle L_1, \models_{L_1}\rangle \leq \langle L_2, \models_{L_2}\rangle$, if for each language $\mathcal{L}$ and sentence $\alpha \in L_1(\mathcal{L})$ there is a sentence $\beta \in L_2(\mathcal{L})$ such that $\mathrm{Mod}_{L_1}^{\mathcal{L}}(\alpha) = \mathrm{Mod}_{L_2}^{\mathcal{L}}(\beta)$. The logics $\langle L_1, \models_{L_1}\rangle$ and $\langle L_2, \models_{L_2}\rangle$ are *equivalent* if $\langle L_1, \models_{L_1}\rangle \leq \langle L_2, \models_{L_2}\rangle$ and $\langle L_2, \models_{L_2}\rangle \leq \langle L_1, \models_{L_1}\rangle$.

Remark 1. First-order logic, i.e., the abstract logic $\langle F, \models\rangle$, is normal. In fact, the above properties are mostly straightforward for first-order logic. We just remark that the expansion property comes down to extensionality, and that the relativization of a sentence α to $R(x, c_1, \dots, c_n)$ is obtained by replacing each subformula $\forall x\, \beta$ by $\forall x\, (R(x, c_1, \dots, c_n) \rightarrow \beta)$. Moreover, if $\langle L, \models_L\rangle$ is normal, then $\langle F, \models\rangle \leq \langle L, \models_L\rangle$, as can be can shown by induction on first-order formulas. Accordingly, with no loss in generality, we can assume that every first-order sentence belongs to every normal logic.

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Bibliography