Definition lin.1. An abstract logic is a pair \( \langle L, |=_L \rangle \), where \( L \) is a function that assigns to each language \( L \) a set \( L(L) \) of sentences, and \( |=_L \) is a relation between structures for the language \( L \) and elements of \( L(L) \). In particular, \( \langle F, |= \rangle \) is ordinary first-order logic, i.e., \( F \) is the function assigning to the language \( L \) the set of first-order sentences built from the constants in \( L \), and \( |= \) is the satisfaction relation of first-order logic.

Notice that we are still employing the same notion of structure for a given language as for first-order logic, but we do not presuppose that sentences are build up from the basic symbols in \( L \) in the usual way, nor that the relation \( |=_L \) is recursively defined in the same way as for first-order logic. So for instance the definition, being completely general, is intended to capture the case where sentences in \( \langle L, |=_L \rangle \) contain infinitely long conjunctions or disjunction, or quantifiers other than \( \exists \) and \( \forall \) (e.g., “there are infinitely many \( x \) such that . . .”), or perhaps infinitely long quantifier prefixes. To emphasize that “sentences” in \( L(L) \) need not be ordinary sentences of first-order logic, in this chapter we use variables \( \alpha, \beta, \ldots \) to range over them, and reserve \( \varphi, \psi, \ldots \) for ordinary first-order formulas.

Definition lin.2. Let \( \text{Mod}_L(\alpha) \) denote the class \( \{ M : M |=_L \alpha \} \). If the language needs to be made explicit, we write \( \text{Mod}^L_L(\alpha) \). Two structures \( M \) and \( N \) for \( L \) are elementarily equivalent in \( \langle L, |=_L \rangle \), written \( M \equiv L N \), if the same sentences from \( L(L) \) are true in each.

Definition lin.3. An abstract logic \( \langle L, |=_L \rangle \) for the language \( L \) is normal if it satisfies the following properties:

1. (\( L \)-Monotonicity) For languages \( L \) and \( L' \), if \( L \subseteq L' \), then \( L(L) \subseteq L(L') \).
2. (Expansion Property) For each \( \alpha \in L(L) \) there is a finite subset \( L' \) of \( L \) such that the relation \( M |=_L \alpha \) depends only on the reduct of \( M \) to \( L' \); i.e., if \( M \) and \( N \) have the same reduct to \( L' \) then \( M |=_L \alpha \) if and only if \( N |=_L \alpha \).
3. (Isomorphism Property) If \( M |=_L \alpha \) and \( M \simeq N \) then also \( N |=_L \alpha \).
4. (Renaming Property) The relation \( |=_L \) is preserved under renaming: if the language \( L' \) is obtained from \( L \) by replacing each symbol \( P \) by a symbol \( P' \) of the same arity and each constant \( c \) by a distinct constant \( c' \), then for each structure \( M \) and sentence \( \alpha \), \( M |=_L \alpha \) if and only if \( M' |=_L \alpha' \), where \( M' \) is the \( L' \)-structure corresponding to \( L \) and \( \alpha' \in L(L') \).
5. (Boolean Property) The abstract logic \( \langle L, |=_L \rangle \) is closed under the Boolean connectives in the sense that for each \( \alpha \in L(L) \) there is a \( \beta \in L(L) \) such that \( \beta \) if and only if \( \neg \alpha \), and for each \( \alpha \) and \( \beta \) there is a \( \gamma \)
such that \( \text{Mod}_L(\gamma) = \text{Mod}_L(\alpha) \cap \text{Mod}_L(\beta) \). Similarly for atomic formulas and the other connectives.

6. (Quantifier Property) For each constant \( c \) in \( L \) and \( \alpha \in L(\mathcal{L}) \) there is a \( \beta \in L(\mathcal{L}) \) such that

\[
\text{Mod}_L' = \{ \mathcal{M} : (\mathcal{M}, a) \in \text{Mod}_L(\alpha) \text{ for some } a \in |\mathcal{M}| \},
\]

where \( L' = L \setminus \{ c \} \) and \( (\mathcal{M}, a) \) is the expansion of \( \mathcal{M} \) to \( L \) assigning \( a \) to \( c \).

7. (Relativization Property) Given a sentence \( \alpha \in L(\mathcal{L}) \) and symbols \( R, c_1, \ldots, c_n \) not in \( L \), there is a sentence \( \beta \in L(\mathcal{L} \cup \{ R, c_1, \ldots, c_n \}) \) called the relativization of \( \alpha \) to \( R(x, c_1, \ldots, c_n) \), such that for each structure \( \mathcal{M} : (\mathcal{M}, X, b_1, \ldots, b_n) \models L \beta \) if and only if \( \mathcal{M} \models L \alpha \), where \( \mathcal{M} \) is the substructure of \( \mathcal{M} \) with domain \( |\mathcal{M}| = \{ a \in |\mathcal{M}| : R(a, b_1, \ldots, b_n) \} \) (see ??), and \( (\mathcal{M}, X, b_1, \ldots, b_n) \) is the expansion of \( \mathcal{M} \) interpreting \( R, c_1, \ldots, c_n \) by \( X, b_1, \ldots, b_n \), respectively (with \( X \subseteq M^{n+1} \)).

Definition lin.4. Given two abstract logics \( \langle L_1, \models_{L_1} \rangle \) and \( \langle L_2, \models_{L_2} \rangle \) we say that the latter is at least as expressive as the former, written \( \langle L_1, \models_{L_1} \rangle \leq \langle L_2, \models_{L_2} \rangle \), if for each language \( \mathcal{L} \) and sentence \( \alpha \in L_1(\mathcal{L}) \) there is a sentence \( \beta \in L_2(\mathcal{L}) \) such that \( \text{Mod}_{L_1}(\alpha) = \text{Mod}_{L_2}(\beta) \). The logics \( \langle L_1, \models_{L_1} \rangle \) and \( \langle L_2, \models_{L_2} \rangle \) are equivalent if \( \langle L_1, \models_{L_1} \rangle \leq \langle L_2, \models_{L_2} \rangle \) and \( \langle L_2, \models_{L_2} \rangle \leq \langle L_1, \models_{L_1} \rangle \).

Remark 1. First-order logic, i.e., the abstract logic \( \langle F, \models \rangle \), is normal. In fact, the above properties are mostly straightforward for first-order logic. We just remark that the expansion property comes down to extensionality, and that the relativization of a sentence \( \alpha \) to \( R(x, c_1, \ldots, c_n) \) is obtained by replacing each subformula \( \forall x \beta \) by \( \forall x (R(x, c_1, \ldots, c_n) \rightarrow \beta) \). Moreover, if \( \langle L, \models_L \rangle \) is normal, then \( \langle F, \models \rangle \leq \langle L, \models_L \rangle \), as can be can shown by induction on first-order formulas. Accordingly, with no loss in generality, we can assume that every first-order sentence belongs to every normal logic.

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Bibliography