

lin.1 Abstract Logics

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Definition lin.1. An *abstract logic* is a pair $\langle L, \models_L \rangle$, where L is a function that assigns to each **language** \mathcal{L} a set $L(\mathcal{L})$ of **sentences**, and \models_L is a relation between **structures** for the **language** \mathcal{L} and **elements** of $L(\mathcal{L})$. In particular, $\langle F, \models \rangle$ is ordinary first-order logic, i.e., F is the function assigning to the **language** \mathcal{L} the set of first-order **sentences** built from the constants in \mathcal{L} , and \models is the satisfaction relation of first-order logic.

Notice that we are still employing the same notion of **structure** for a given **language** as for first-order logic, but we do not presuppose that **sentences** are built up from the basic symbols in \mathcal{L} in the usual way, nor that the relation \models_L is recursively defined in the same way as for first-order logic. So for instance the definition, being completely general, is intended to capture the case where **sentences** in $\langle L, \models_L \rangle$ contain infinitely long conjunctions or disjunction, or quantifiers other than \exists and \forall (e.g., “there are infinitely many x such that ...”), or perhaps infinitely long quantifier prefixes. To emphasize that “**sentences**” in $L(\mathcal{L})$ need not be ordinary **sentences** of first-order logic, in this chapter we use **variables** α, β, \dots to range over them, and reserve φ, ψ, \dots for ordinary first-order **formulas**.

Definition lin.2. Let $\text{Mod}_L(\alpha)$ denote the class $\{\mathfrak{M} : \mathfrak{M} \models_L \alpha\}$. If the **language** needs to be made explicit, we write $\text{Mod}_L^{\mathcal{L}}(\alpha)$. Two **structures** \mathfrak{M} and \mathfrak{N} for \mathcal{L} are *elementarily equivalent in* $\langle L, \models_L \rangle$, written $\mathfrak{M} \equiv_L \mathfrak{N}$, if the same **sentences** from $L(\mathcal{L})$ are true in each.

Definition lin.3. An abstract logic $\langle L, \models_L \rangle$ for the **language** \mathcal{L} is *normal* if it satisfies the following properties:

1. (*L-Monotonicity*) For **languages** \mathcal{L} and \mathcal{L}' , if $\mathcal{L} \subseteq \mathcal{L}'$, then $L(\mathcal{L}) \subseteq L(\mathcal{L}')$.
2. (*Expansion Property*) For each $\alpha \in L(\mathcal{L})$ there is a *finite* subset \mathcal{L}' of \mathcal{L} such that the relation $\mathfrak{M} \models_L \alpha$ depends only on the reduct of \mathfrak{M} to \mathcal{L}' ; i.e., if \mathfrak{M} and \mathfrak{N} have the same reduct to \mathcal{L}' then $\mathfrak{M} \models_L \alpha$ if and only if $\mathfrak{N} \models_L \alpha$.
3. (*Isomorphism Property*) If $\mathfrak{M} \models_L \alpha$ and $\mathfrak{M} \simeq \mathfrak{N}$ then also $\mathfrak{N} \models_L \alpha$.
4. (*Renaming Property*) The relation \models_L is preserved under renaming: if the **language** \mathcal{L}' is obtained from \mathcal{L} by replacing each symbol P by a symbol P' of the same arity and each constant c by a distinct constant c' , then for each **structure** \mathfrak{M} and **sentence** α , $\mathfrak{M} \models_L \alpha$ if and only if $\mathfrak{M}' \models_L \alpha'$, where \mathfrak{M}' is the \mathcal{L}' -**structure** corresponding to \mathcal{L} and $\alpha' \in L(\mathcal{L}')$.
5. (*Boolean Property*) The abstract logic $\langle L, \models_L \rangle$ is closed under the Boolean connectives in the sense that for each $\alpha \in L(\mathcal{L})$ there is a $\beta \in L(\mathcal{L})$ such that $\mathfrak{M} \models_L \beta$ if and only if $\mathfrak{M} \not\models_L \alpha$, and for each α and β there is a γ

such that $\text{Mod}_L(\gamma) = \text{Mod}_L(\alpha) \cap \text{Mod}_L(\beta)$. Similarly for atomic formulas and the other connectives.

6. (*Quantifier Property*) For each constant c in \mathcal{L} and $\alpha \in L(\mathcal{L})$ there is a $\beta \in L(\mathcal{L})$ such that

$$\text{Mod}_L^{\mathcal{L}'}(\beta) = \{\mathfrak{M} : (\mathfrak{M}, a)\} \in \text{Mod}_L^{\mathcal{L}}(\alpha) \text{ for some } a \in |\mathfrak{M}|\},$$

where $\mathcal{L}' = \mathcal{L} \setminus \{c\}$ and (\mathfrak{M}, a) is the expansion of \mathfrak{M} to \mathcal{L} assigning a to c .

7. (*Relativization Property*) Given a sentence $\alpha \in L(\mathcal{L})$ and symbols R, c_1, \dots, c_n not in \mathcal{L} , there is a sentence $\beta \in L(\mathcal{L} \cup \{R, c_1, \dots, c_n\})$ called the *relativization* of α to $R(x, c_1, \dots, c_n)$, such that for each structure \mathfrak{M} :

$$(\mathfrak{M}, X, b_1, \dots, b_n) \models_L \beta \text{ if and only if } \mathfrak{N} \models_L \alpha,$$

where \mathfrak{N} is the substructure of \mathfrak{M} with domain $|\mathfrak{N}| = \{a \in |\mathfrak{M}| : R^{\mathfrak{M}}(a, b_1, \dots, b_n)\}$ (see ??), and $(\mathfrak{M}, X, b_1, \dots, b_n)$ is the expansion of \mathfrak{M} interpreting R, c_1, \dots, c_n by X, b_1, \dots, b_n , respectively (with $X \subseteq M^{n+1}$).

Definition lin.4. Given two abstract logics $\langle L_1, \models_{L_1} \rangle$ and $\langle L_2, \models_{L_2} \rangle$ we say that the latter is *at least as expressive* as the former, written $\langle L_1, \models_{L_1} \rangle \leq \langle L_2, \models_{L_2} \rangle$, if for each language \mathcal{L} and sentence $\alpha \in L_1(\mathcal{L})$ there is a sentence $\beta \in L_2(\mathcal{L})$ such that $\text{Mod}_{L_1}^{\mathcal{L}}(\alpha) = \text{Mod}_{L_2}^{\mathcal{L}}(\beta)$. The logics $\langle L_1, \models_{L_1} \rangle$ and $\langle L_2, \models_{L_2} \rangle$ are *equivalent* if $\langle L_1, \models_{L_1} \rangle \leq \langle L_2, \models_{L_2} \rangle$ and $\langle L_2, \models_{L_2} \rangle \leq \langle L_1, \models_{L_1} \rangle$.

Remark 1. First-order logic, i.e., the abstract logic $\langle F, \models \rangle$, is normal. In fact, the above properties are mostly straightforward for first-order logic. We just remark that the expansion property comes down to extensionality, and that the relativization of a sentence α to $R(x, c_1, \dots, c_n)$ is obtained by replacing each subformula $\forall x \beta$ by $\forall x (R(x, c_1, \dots, c_n) \rightarrow \beta)$. Moreover, if $\langle L, \models_L \rangle$ is normal, then $\langle F, \models \rangle \leq \langle L, \models_L \rangle$, as can be shown by induction on first-order formulas. Accordingly, with no loss in generality, we can assume that every first-order sentence belongs to every normal logic.

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Bibliography