int.1 Separation of Sentences

mod:int:sep:sec A bit of groundwork is needed before we can proceed with the proof of the interpolation theorem. An interpolant for φ and ψ is a sentence χ such that $\varphi \vDash \chi$ and $\chi \vDash \psi$. By contraposition, the latter is true iff $\neg\psi \vDash \neg\chi$. A sentence χ with this property is said to *separate* φ and $\neg\psi$. So finding an interpolant for φ and ψ amounts to finding a sentence that separates φ and $\neg\psi$. As so often, it will be useful to consider a generalization: a sentence that separates two *sets* of sentences.

Definition int.1. A sentence χ *separates* sets of sentences Γ and Δ if and only if $\Gamma \vDash \chi$ and $\Delta \vDash \neg\chi$. If no such sentence exists, then Γ and Δ are *inseparable*.

The inclusion relations between the classes of models of Γ, Δ and χ are represented below:

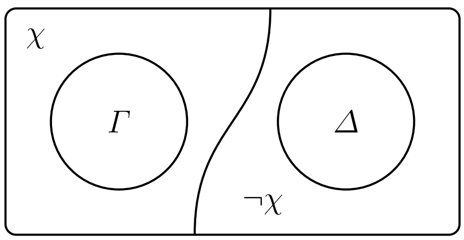

Figure 1: χ separates Γ and Δ

mod:int:sep:fig:sep mod:int:sep:lem:sep1 **Lemma int.2.** *Suppose $\mathcal{L}_0$ is the language containing every constant symbol, function symbol and predicate symbol (other than $\doteq$) that occurs in* both *Γ and Δ, and let $\mathcal{L}'_0$ be obtained by the addition of infinitely many new constant symbols c_n for $n \geq 0$. Then if Γ and Δ are inseparable in $\mathcal{L}_0$, they are also inseparable in $\mathcal{L}'_0$.*

Proof. We proceed indirectly: suppose by way of contradiction that Γ and Δ are separated in $\mathcal{L}'_0$. Then $\Gamma \vDash \chi[c/x]$ and $\Delta \vDash \neg\chi[c/x]$ for some $\chi \in \mathcal{L}_0$ (where c is a new constant symbol—the case where χ contains more than one such new constant symbol is similar). By compactness, there are *finite* subsets Γ_0 of Γ and Δ_0 of Δ such that $\Gamma_0 \vDash \chi[c/x]$ and $\Delta_0 \vDash \neg\chi[c/x]$. Let γ be the conjunction of all formulas in Γ_0 and δ the conjunction of all formulas in Δ_0. Then

$$\gamma \vDash \chi[c/x], \qquad \delta \vDash \neg\chi[c/x].$$

From the former, by Generalization, we have $\gamma \vDash \forall x\, \chi$, and from the latter by contraposition, $\chi[c/x] \vDash \neg\delta$, whence also $\forall x\, \chi \vDash \neg\delta$. Contraposition again gives $\delta \vDash \neg\forall x\, \chi$. By monotonicity,

$$\Gamma \vDash \forall x\, \chi, \qquad \Delta \vDash \neg\forall x\, \chi,$$

so that $\forall x\, \chi$ separates Γ and Δ in $\mathcal{L}_0$. □

Lemma int.3. *Suppose that $\Gamma \cup \{\exists x\, \sigma\}$ and Δ are inseparable, and c is a new constant symbol not in Γ, Δ, or σ. Then $\Gamma \cup \{\exists x\, \sigma, \sigma[c/x]\}$ and Δ are also inseparable.* mod:int:sep: lem:sep2

Proof. Suppose for contradiction that χ separates $\Gamma \cup \{\exists x\, \sigma, \sigma[c/x]\}$ and Δ, while at the same time $\Gamma \cup \{\exists x \sigma\}$ and Δ are inseparable. We distinguish two cases:

1. c does not occur in χ: in this case $\Gamma \cup \{\exists x\, \sigma, \neg\chi\}$ is satisfiable (otherwise χ separates $\Gamma \cup \{\exists x\, \sigma\}$ and Δ). It remains so if $\sigma[c/x]$ is added, so χ does not separate $\Gamma \cup \{\exists x\, \sigma, \sigma[c/x]\}$ and Δ after all.

2. c does occur in χ so that χ has the form $\chi[c/x]$. Then we have that

$$\Gamma \cup \{\exists x\, \sigma, \sigma[c/x]\} \vDash \chi[c/x],$$

whence $\Gamma, \exists x\, \sigma \vDash \forall x\, (\sigma \to \chi)$ by the Deduction Theorem and Generalization, and finally $\Gamma \cup \{\exists x\, \sigma\} \vDash \exists x\, \chi$. On the other hand, $\Delta \vDash \neg\chi[c/x]$ and hence by Generalization $\Delta \vDash \neg\exists x\, \chi$. So $\Gamma \cup \{\exists x\, \sigma\}$ and Δ are separable, a contradiction. □

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Bibliography