int.1 The Definability Theorem

mod:int:def:sec One important application of the interpolation theorem is Beth's definability theorem. To define an n-place relation R we can give a formula χ with n free variables which does not involve R. This would be an *explicit* definition of R in terms of χ. We can then say also that a theory $\Sigma(P)$ in a language containing the n-place predicate symbol P explicitly defines P if it contains (or at least entails) a formalized explicit definition, i.e.,

$$\Sigma(P) \vDash \forall x_1 \dots \forall x_n \, (P(x_1, \dots, x_n) \leftrightarrow \chi(x_1, \dots, x_n)).$$

But an explicit definition is only one way of defining—in the sense of determining completely—a relation. A theory may also be such that the interpretation of P is fixed by the interpretation of the rest of the language in any model. The definability theorem states that whenever a theory fixes the interpretation of P in this way—whenever it *implicitly defines* P—then it also explicitly defines it.

Definition int.1. Suppose $\mathcal{L}$ is a language not containing the predicate symbol P. A set $\Sigma(P)$ of sentences of $\mathcal{L} \cup \{P\}$ *explicitly defines* P if and only if there is a formula $\chi(x_1, \dots, x_n)$ of $\mathcal{L}$ such that

$$\Sigma(P) \vDash \forall x_1 \dots \forall x_n \, (P(x_1, \dots, x_n) \leftrightarrow \chi(x_1, \dots, x_n)).$$

Definition int.2. Suppose $\mathcal{L}$ is a language not containing the predicate symbols P and P'. A set $\Sigma(P)$ of sentences of $\mathcal{L} \cup \{P\}$ *implicitly defines* P if and only if

$$\Sigma(P) \cup \Sigma(P') \vDash \forall x_1 \dots \forall x_n \, (P(x_1, \dots, x_n) \leftrightarrow P'(x_1, \dots, x_n)),$$

where $\Sigma(P')$ is the result of uniformly replacing P with P' in $\Sigma(P)$.

In other words, for any model $\mathfrak{M}$ and $R, R' \subseteq |\mathfrak{M}|^n$, if both $(\mathfrak{M}, R) \vDash \Sigma(P)$ and $(\mathfrak{M}, R') \vDash \Sigma(P')$, then $R = R'$; where $(\mathfrak{M}, R)$ is the structure $\mathfrak{M}'$ for the expansion of $\mathcal{L}$ to $\mathcal{L} \cup \{P\}$ such that $P^{\mathfrak{M}'} = R$, and similarly for $(\mathfrak{M}, R')$.

Theorem int.3 (Beth Definability Theorem). *A set $\Sigma(P)$ of $\mathcal{L} \cup \{P\}$-formulas implicitly defines P if and only $\Sigma(P)$ explicitly defines P.*

Proof. If $\Sigma(P)$ explicitly defines P then both

$$\begin{aligned} \Sigma(P) &\vDash \quad \forall x_1 \dots \forall x_n \, (P(x_1, \dots, x_n) \leftrightarrow \chi(x_1, \dots, x_n)) \\ \Sigma(P') &\vDash \quad \forall x_1 \dots \forall x_n \, (P'(x_1, \dots, x_n) \leftrightarrow \chi(x_1, \dots, x_n)) \end{aligned}$$

and the conclusion follows. For the converse: assume that $\Sigma(P)$ implicitly defines P. First, we add constant symbols $c_1, \dots, c_n$ to $\mathcal{L}$. Then

$$\Sigma(P) \cup \Sigma(P') \vDash P(c_1, \dots, c_n) \to P'(c_1, \dots, c_n).$$

By compactness, there are finite sets $\Delta_0 \subseteq \Sigma(P)$ and $\Delta_1 \subseteq \Sigma(P')$ such that

$$\Delta_0 \cup \Delta_1 \vDash P(c_1, \dots, c_n) \to P'(c_1, \dots, c_n).$$

Let $\theta(P)$ be the conjunction of all sentences $\varphi(P)$ such that either $\varphi(P) \in \Delta_0$ or $\varphi(P') \in \Delta_1$ and let $\theta(P')$ be the conjunction of all sentences $\varphi(P')$ such that either $\varphi(P) \in \Delta_0$ or $\varphi(P') \in \Delta_1$. Then $\theta(P) \wedge \theta(P') \vDash P(c_1, \dots, c_n) \to P' c_1 \dots c_n$. We can re-arrange this so that each predicate symbol occurs on one side of $\vDash$:

$$\theta(P) \wedge P(c_1, \dots, c_n) \vDash \theta(P') \to P'(c_1, \dots, c_n).$$

By Craig's Interpolation Theorem there is a sentence $\chi(c_1, \dots, c_n)$ not containing P or P' such that:

$$\theta(P) \wedge P(c_1, \dots, c_n) \vDash \chi(c_1, \dots, c_n); \quad \chi(c_1, \dots, c_n) \vDash \theta(P') \to P'(c_1, \dots, c_n).$$

From the former of these two entailments we have: $\theta(P) \vDash P(c_1, \dots, c_n) \to \chi(c_1, \dots, c_n)$. And from the latter, since an $\mathcal{L} \cup \{P\}$-model $(\mathfrak{M}, R) \vDash \varphi(P)$ if and only if the corresponding $\mathcal{L} \cup \{P'\}$-model $(\mathfrak{M}, R) \models \varphi(P')$, we have $\chi(c_1, \dots, c_n) \vDash \theta(P) \to P(c_1, \dots, c_n)$, from which:

$$\theta(P) \vDash \chi(c_1, \dots, c_n) \to P(c_1, \dots, c_n).$$

Putting the two together, $\theta(P) \vDash P(c_1, \dots, c_n) \leftrightarrow \chi(c_1, \dots, c_n)$, and by monotony and generalization also

$$\Sigma(P) \vDash \forall x_1 \dots \forall x_n \, (P(x_1, \dots, x_n) \leftrightarrow \chi(x_1, \dots, x_n)). \qquad \square$$

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Bibliography